# The Micro-world of Cographs

Bogdan Alecu Vadim Lozin Dominique de Werra

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#### **Basic definitions**

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Let p be a parameter and X a graph class. We say p is bounded in X if there is a constant k such that p(G) ≤ k for all G ∈ X, and unbounded in X otherwise.

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In fact, they are the *only* minimal classes of unbounded  $\nu$ .

Theorem (Ramsey)

A class  $\mathcal{X}$  is of unbounded  $\nu$  if and only if it contains  $\mathcal{K}$  or  $\overline{\mathcal{K}}$ .

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A bypass: restrict ourselves to a setting in which this does not happen.

## Well-quasi-orderability

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#### Well-quasi-orderability

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- The set of all classes of graphs is not wqo (not even well-founded) under inclusion. An infinite strictly descending chain is given by X<sub>i</sub> := Free(C<sub>3</sub>,..., C<sub>i</sub>), i ≥ 3.

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### Corollary

Let  $\mathcal{X}$  be a wqo class,  $\mathcal{Y} \subseteq \mathcal{X}$  a subclass, and let p be a parameter. If p is unbounded in  $\mathcal{Y}$ , then there exists a minimal class  $\mathcal{Z} \subseteq \mathcal{Y}$  of unbounded p.

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In other words, if  $\mathcal{X}$  is wqo, then unboundedness in  $\mathcal{X}$  of any parameter can be characterised in terms of minimal classes.

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#### Theorem (Damaschke)

The class of cographs is wqo under the induced subgraph relation.

P. Damaschke, Induced subgraphs and well-quasi-ordering. *J. Graph Theory* 14(4), 427–435 (1990).

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A class of cographs has unbounded linear clique-width if and only if it contains all quasi-threshold graphs or their complements.

R. Brignall, N. Korpelainen, V. Vatter, Linear clique-width for hereditary classes of cographs. *J. Graph Theory* 84, 501-511 (2017).

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We present several more results of this type: for various parameters that are unbounded in the class of cographs, we find the minimal classes where the parameters are unbounded.

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## Co-chromatic number

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### Definition

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#### Theorem

Let  $\mathcal{U}$  be the class of unions of cliques.  $\mathcal{U}$  and  $\overline{\mathcal{U}}$  are the only minimal hereditary subclasses of cographs of unbounded cochromatic number.

# Lettericity (Petkovšek, 2002)

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#### Theorem

Let  $\mathcal{M}$  be the class of graphs of vertex degree at most 1.  $\mathcal{M}$  and  $\overline{\mathcal{M}}$  are the only minimal hereditary subclasses of cographs of unbounded lettericity.
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#### Theorem

 $\overline{\mathcal{M}}$  is the only minimal hereditary subclass of cographs of unbounded boxicity.

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#### Theorem

Let  $\mathcal{B}$  be the class of complete bipartite graphs, and  $\mathcal{F}$  the class of star forests.  $\mathcal{K}, \mathcal{B}$  and  $\mathcal{F}$  are the only minimal hereditary subclasses of cographs of unbounded *H*-index.

A complete k-colouring of a graph G is a partition of G into k independent sets such that any two of the independent sets have an edge between them. The achromatic number of G is the maximum k such that G admits a complete k-colouring.

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#### Theorem

 ${\cal K}$  and  ${\cal M}$  are the only minimal hereditary subclasses of cographs of unbounded achromatic number.

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#### Definition

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#### Proposition

 $Free(S) \subseteq Free(T)$  if and only if every graph  $H \in T$  has a graph  $G \in S$  with  $G \leq H$ .

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Write M(p), M(q) for the respective sets of minimal classes of cographs where p and q are unbounded. p is stronger than q if and only if for every class  $\mathcal{Y} \in M(q)$  there is a class  $\mathcal{X} \in M(p)$  with  $\mathcal{X} \subseteq \mathcal{Y}$ .

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Figure: A Hasse diagram of graph parameters within the universe of cographs

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This helps, but it is not a complete answer.

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# Thank you!

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