

# On the Complexity of the Singly Connected Vertex Deletion.

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## Singly Connected Digraphs

- ▶ **Singly Connected Digraphs:** For every pair  $u$  and  $v$  of digraph  $D$ , at most one (directed) path from  $u$  to  $v$ .

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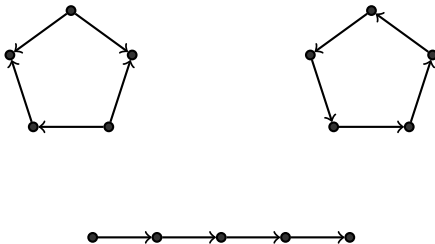


Figure: Examples of Singly Connected Digraphs.

# Singly Connected Vertex Deletion problem

## Obstruction Structure

### Lemma 1

*Digraph  $D$  is not singly connected  $\iff \exists$  two vertices  $u, v$  with two internally vertex disjoint paths from  $u$  to  $v$ .*

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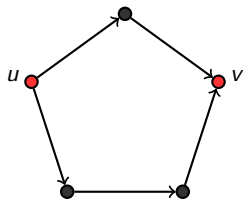


Figure: Obstruction of SCVD

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- ▶ Input
  - ▶ digraph  $D$
  - ▶ positive integer  $k$
- ▶ Problem
  - ▶ Does there exist  $S \subseteq V(D)$  such that  $|S| \leq k$  and  $D \setminus S$  is singly connected?

# Main Results

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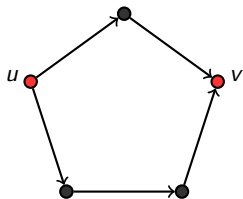
- ▶ Singly connectivity in undirected graphs means forests
- ▶ 2 versions of FVS in undirected graphs:
  - ▶ Deletion to cycle free graphs: DFVS
  - ▶ Deletion to singly connected graphs: SCVD

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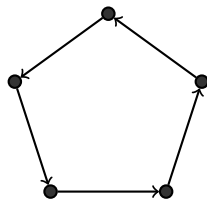
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(a) Obstruction of SCVD



(b) Obstruction of DFVS

# Singly Connected Vertex Deletion problem

Related Work

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- ▶ Recognition of Singly Connected digraphs:  $O(s.t + m)$  (Dietzfelbinger and Jaber).



# Singly Connected Vertex Deletion problem

## Related Work

- ▶ Recognition of Singly Connected digraphs:  $O(s.t + m)$  (Dietzfelbinger and Jaber).
- ▶ NP Completeness of SCVD in digraphs: Dietzfelbinger and Jaber.

# SCVD in $\alpha$ -bounded digraphs

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$\alpha$ -bounded digraphs

$\alpha$ -bounded digraphs are digraphs whose maximum independent set size is bounded by  $\alpha$ .

# SCVD in $\alpha$ -bounded digraphs

## Tournaments

### Lemma 2

*Tournament on four or more vertices is not singly connected.*

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- ▶ Algorithm for SCVD in tournaments is implied trivially by the lemma.
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- ▶ Iterate over all three subset induced digraphs.

The runtime of the algorithm is  $O(n^3)$ .



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## Definitions

**Forward arcs.** For a path  $P = v_1 \dots v_l$  in digraph  $D$ ,  $(v_i, v_j)$  is a forward arc with respect to  $P$  if  $j > i + 1$ .

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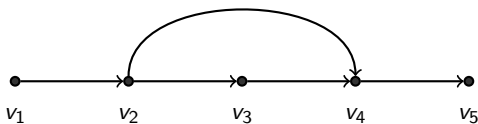


Figure: Forward arcs in a path.

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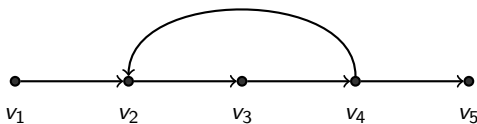


Figure: Backward arcs in a path.

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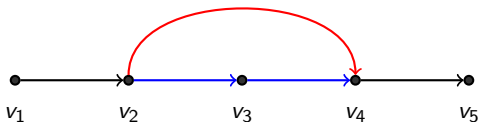
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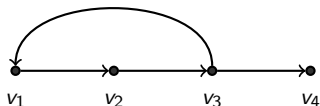
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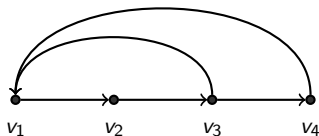


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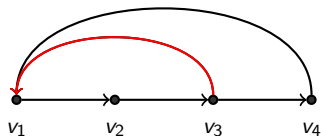


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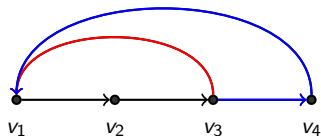


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- ▶ FVS in tournaments: NP hard

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Algo Lemma

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## Algo Lemma

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*For every  $\alpha \in \mathbb{N}$ , every  $\alpha$ -bounded digraph with at least  $\alpha(2\alpha + 4)$  vertices is not singly connected.*

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- ▶ The algorithm for SCVD trivially follows from Algo Lemma. It is exactly like SCVD in tournaments.
- ▶ Runtime:  $O(n^{\alpha(2\alpha+4)-1})$ .

# SCVD in $\alpha$ -bounded digraphs.

## Lemmas

### Theorem 4 (Gallai Milgram)

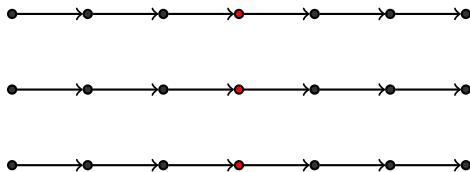
*Every directed graph  $D$  has a path cover  $\mathcal{P}$  and a maximal independent set  $\{v_P \mid P \in \mathcal{P}\}$  such that  $v_P \in P$  for every  $P \in \mathcal{P}$ .*

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### Observation 2

*For a path  $P$  in a singly connected digraph  $D$ ,  $v_P \in V(P)$  can have at most two backward arcs incident on it.*

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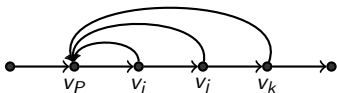
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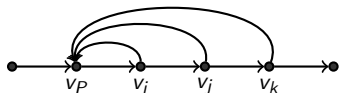
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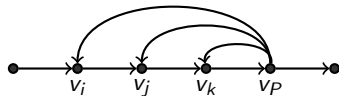
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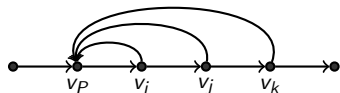
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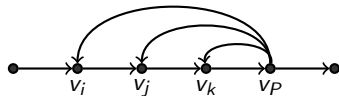
(b) Possibility 2

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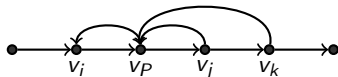
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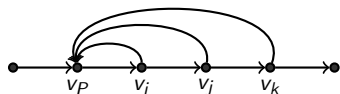


(c) Possibility 3

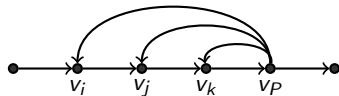


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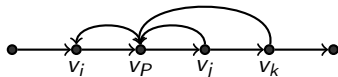
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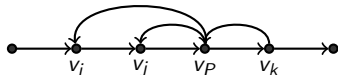
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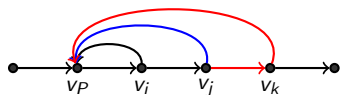
(c) Possibility 3



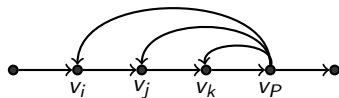
(d) Possibility 4

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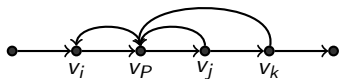
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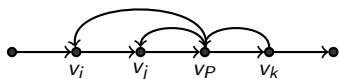
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(b) 2 possible paths from  $v_P$  to  $v_j$



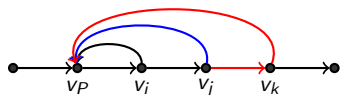
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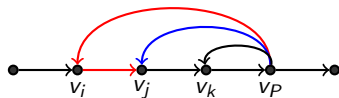
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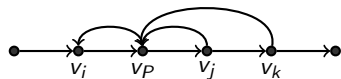
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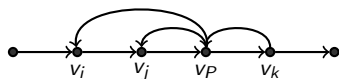
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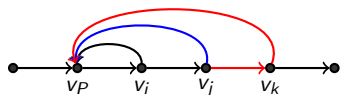
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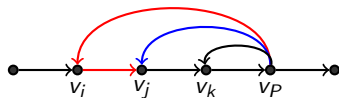
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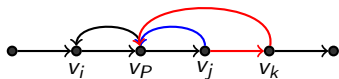
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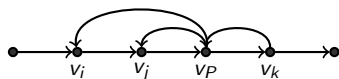
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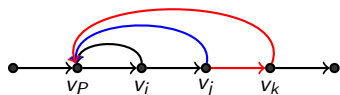
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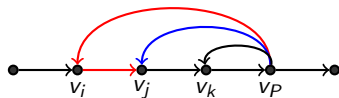
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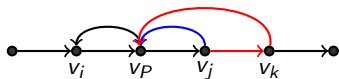
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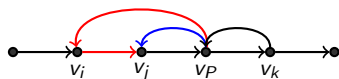
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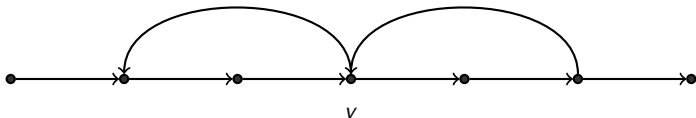


Figure: Only possible configuration for 2 backward arcs incident on  $v$

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### Observation 3

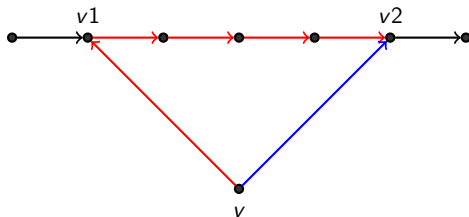
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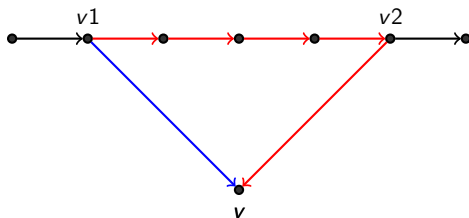


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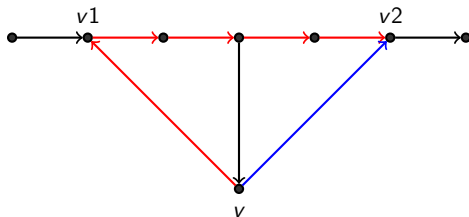


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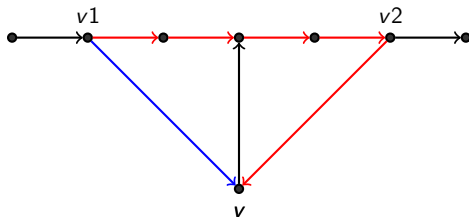


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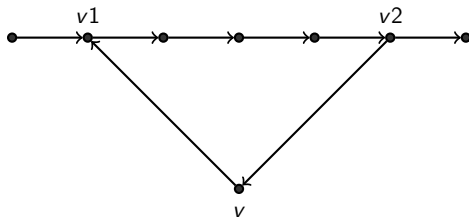


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*For every  $\alpha \in \mathbb{N}$ , every  $\alpha$ -bounded digraph with at least  $\alpha(2\alpha + 4)$  vertices is not singly connected.*

- ▶ Assume not. Let  $D$  be  $\alpha$ -bounded singly connected digraph with  $|V(D)| \geq \alpha(2\alpha + 4)$ .
- ▶ By Gallai Milgram, there is a path cover  $\mathcal{P}$ , and maximal independent set  $I = \{v_P | P \in \mathcal{P}\}$ .

# SCVD in $\alpha$ -bounded digraphs

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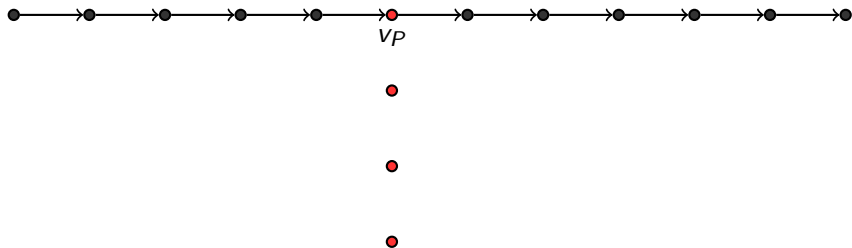
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- ▶ By Gallai Milgram, there is a path cover  $\mathcal{P}$ , and maximal independent set  $I = \{v_P | P \in \mathcal{P}\}$ .
- ▶ By averaging argument, there is a path  $P$  such that  $|V(P)| \geq 2\alpha + 4$ .



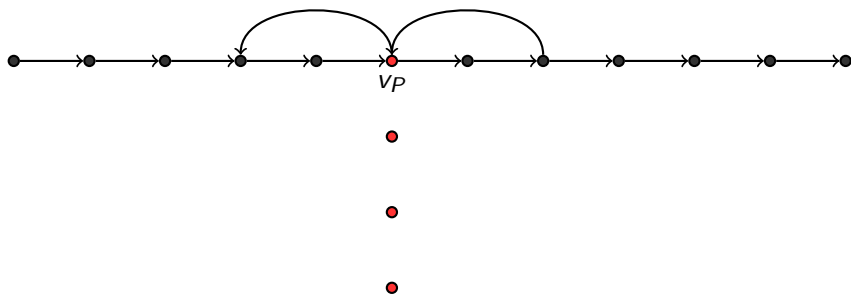
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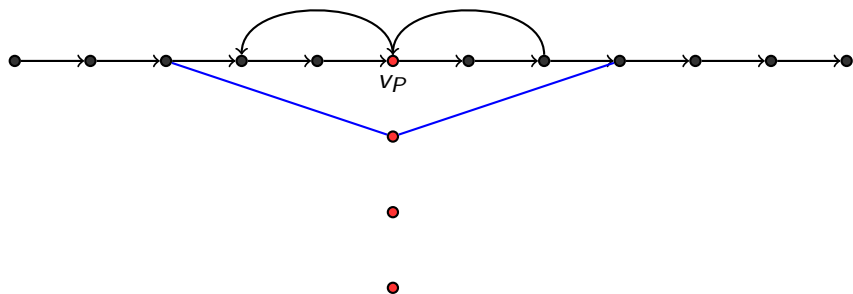
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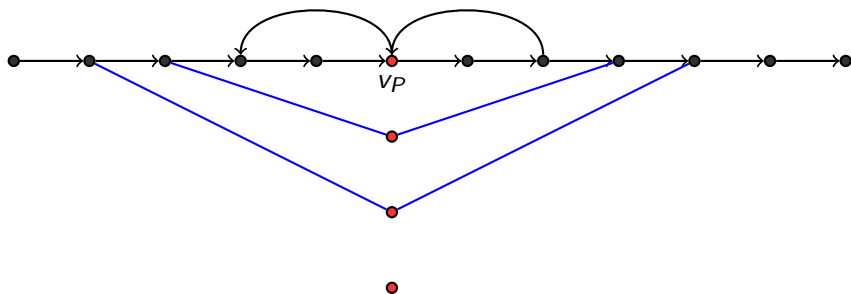
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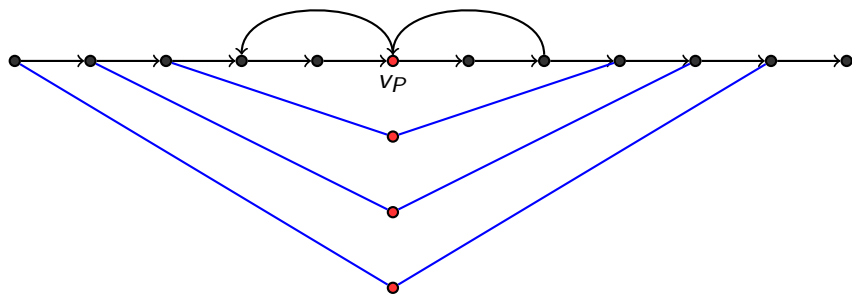
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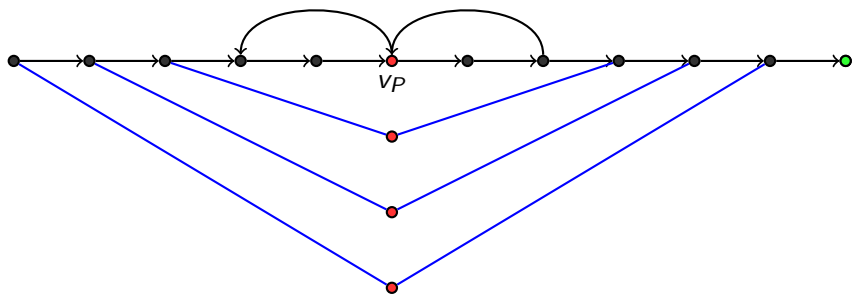
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- ▶ This contradicts our assumption that  $I$  is a maximal independent set.

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- ▶ Is there a  $o(2^n)$  exact algorithm for SCVD in general digraphs?
- ▶ Is SCVD problem in general digraphs parameterized by the solution set size FPT?

Thank You.