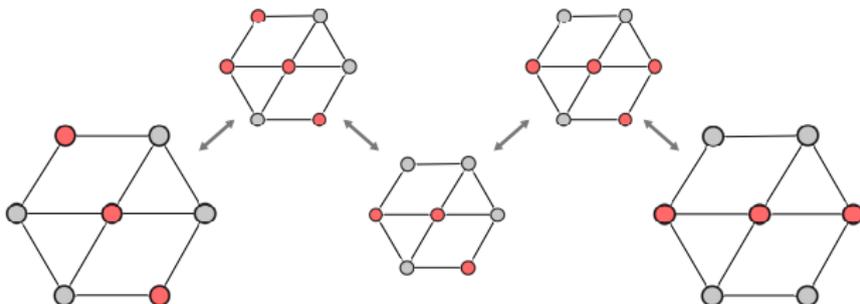


# Decremental Optimization of Dominating Sets Under the Reconfiguration Framework

Alexandre Blanché<sup>1</sup> Haruka Mizuta<sup>2</sup> Paul Ouvrard<sup>1</sup>  
Akira Suzuki<sup>2</sup>

<sup>1</sup>LaBRI, Bordeaux

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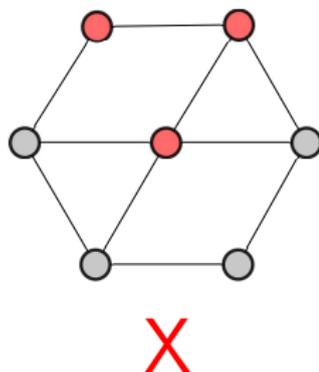
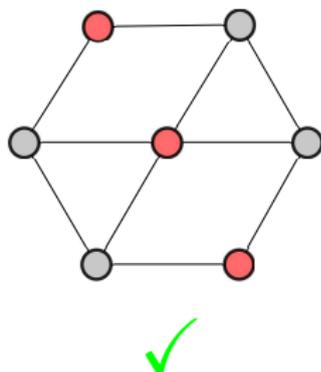


June 10, 2020

# Dominating sets

## Definition

A **dominating set** is a subset of vertices whose neighborhood contains all the vertices.



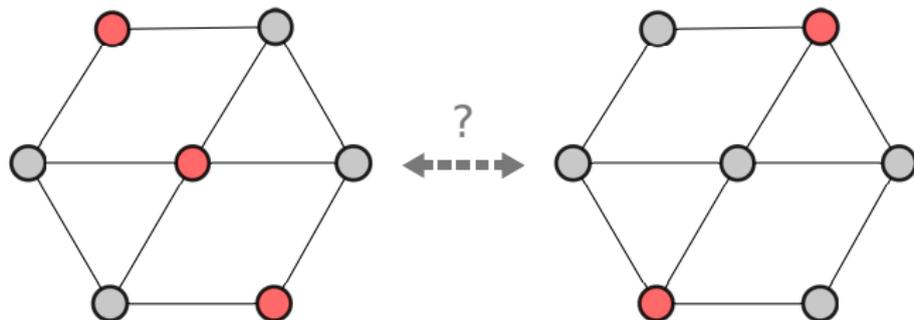
## Dominating set problem

- **Instance** : A graph  $G$ , an integer  $s$
- **Question** : Does  $G$  have a dominating set of size at most  $s$  ?

This problem is NP-complete.

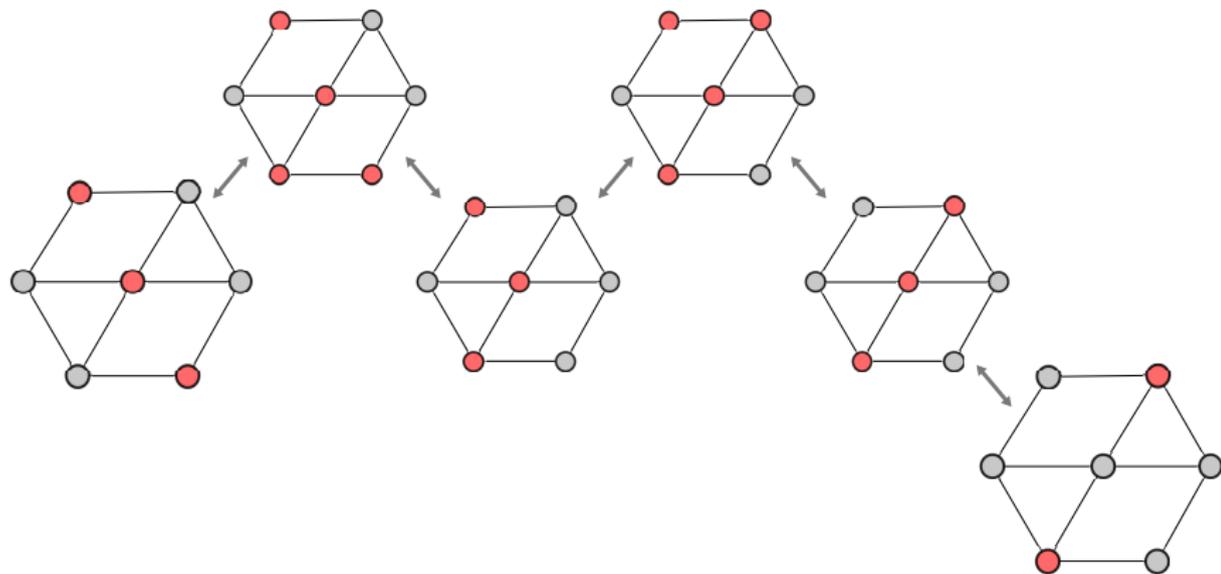
# Dominating set reconfiguration

Model : Successive additions and removals of vertices



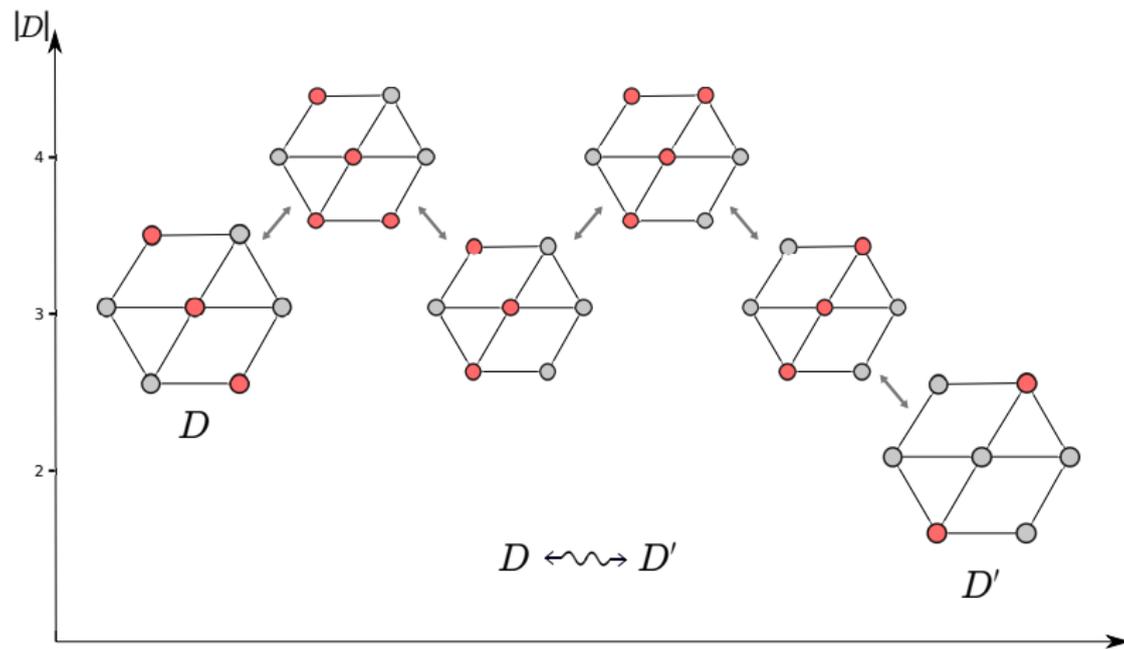
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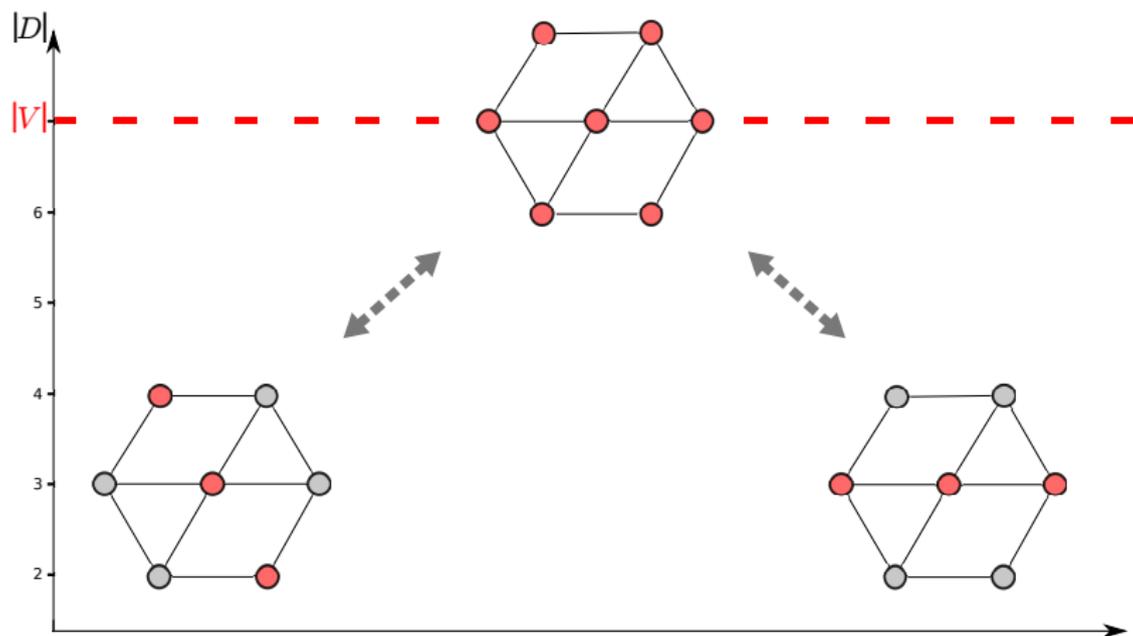
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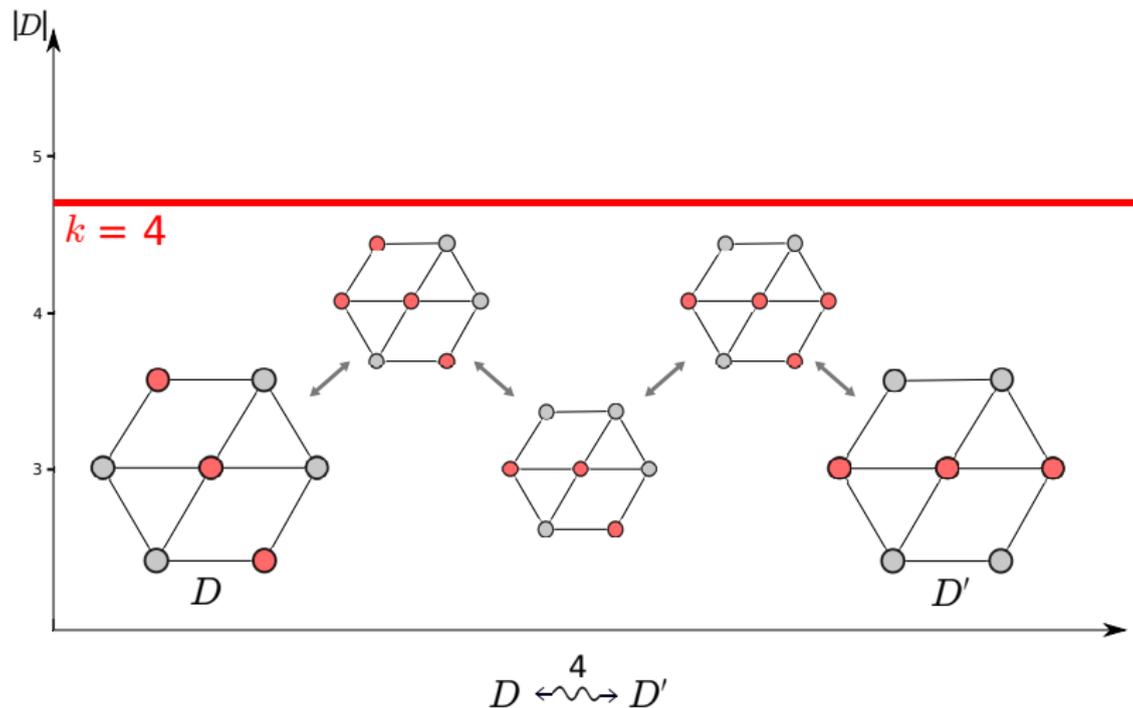
# Size bound

Without a bound on the size of the dominating sets, they are all reachable through successive additions and removals:



# Size bound

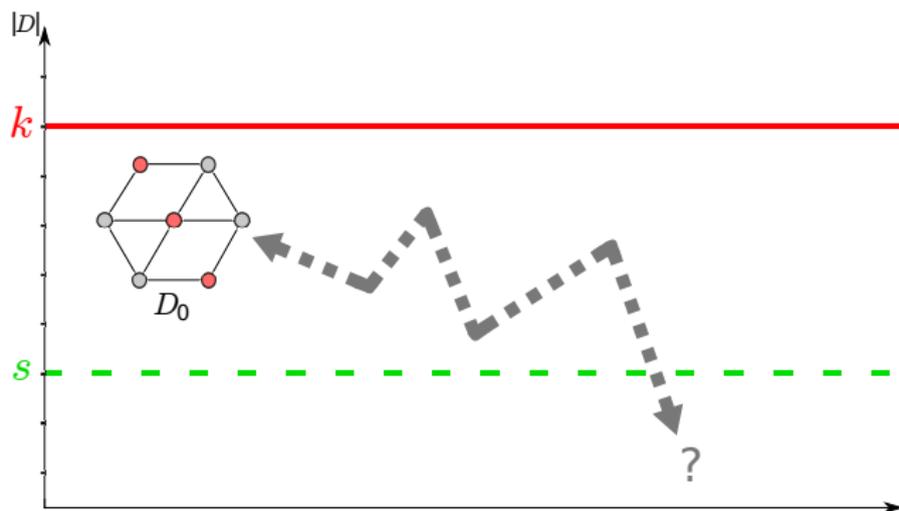
We restrict the size of the authorized dominating sets with a bound  $k$ :



# Optimization problem

## OPT-DSR (OPTimization variant of Dominating Set Reconfiguration)

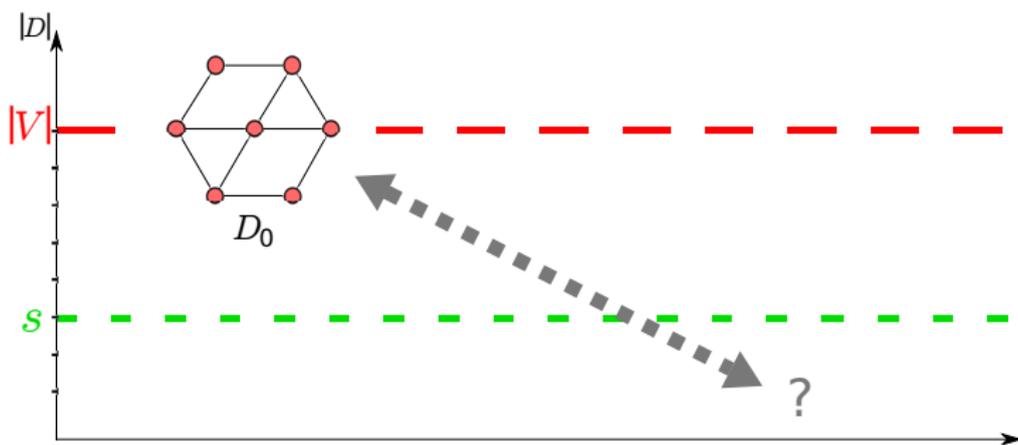
- **Input** : A graph  $G$ , two integers  $k, s$ , a dominating set  $D_0$  of size  $|D_0| \leq k$ .
- **Question** : Does  $G$  have a dominating set  $D_s$  of size  $|D_s| \leq s$ , such that  $D_0 \overset{k}{\leftrightarrow} D_s$  ?



## Observation

OPT-DSR generalizes the dominating set problem.

A graph  $G = (V, E)$  has a dominating set of size  $\leq s$   
 $\Leftrightarrow$   
the instance  $(G, k = |V|, s, D = V)$  is a solution of OPT-DSR.



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A graph  $G = (V, E)$  has a dominating set of size  $\leq s$   
 $\Leftrightarrow$   
the instance  $(G, k = |V|, s, D = V)$  is a solution of OPT-DSR.

## Corollary

OPT-DSR is NP-hard.

$(NP \subseteq PSPACE \subseteq EXPTIME)$

**Theorem [B., Mizuta, Ouvrard, Suzuki (2020)]**

OPT-DSR is PSPACE-complete, including when the input graph :

- is bipartite ;

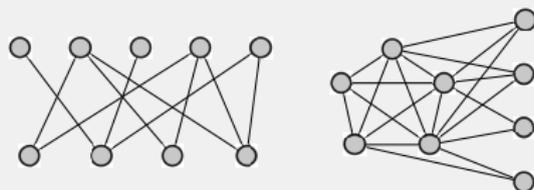


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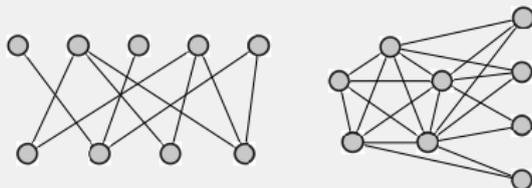


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OPT-DSR is PSPACE-complete, including when the input graph :

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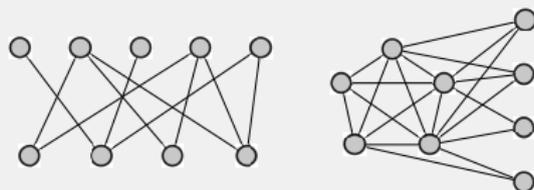


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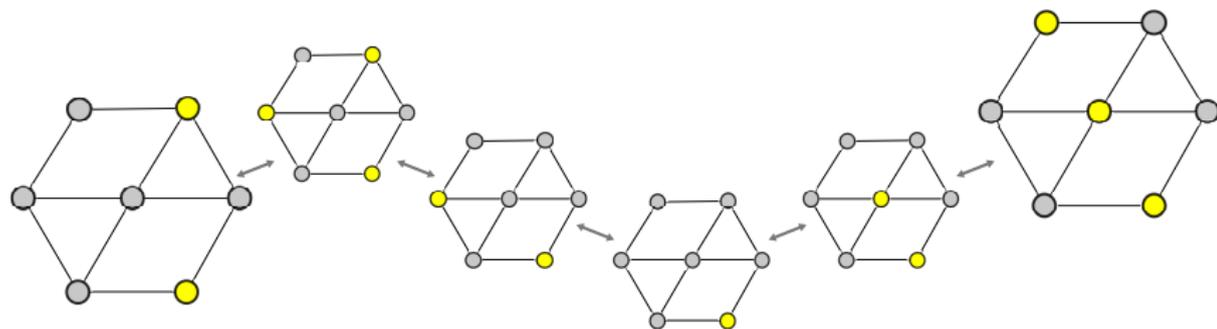
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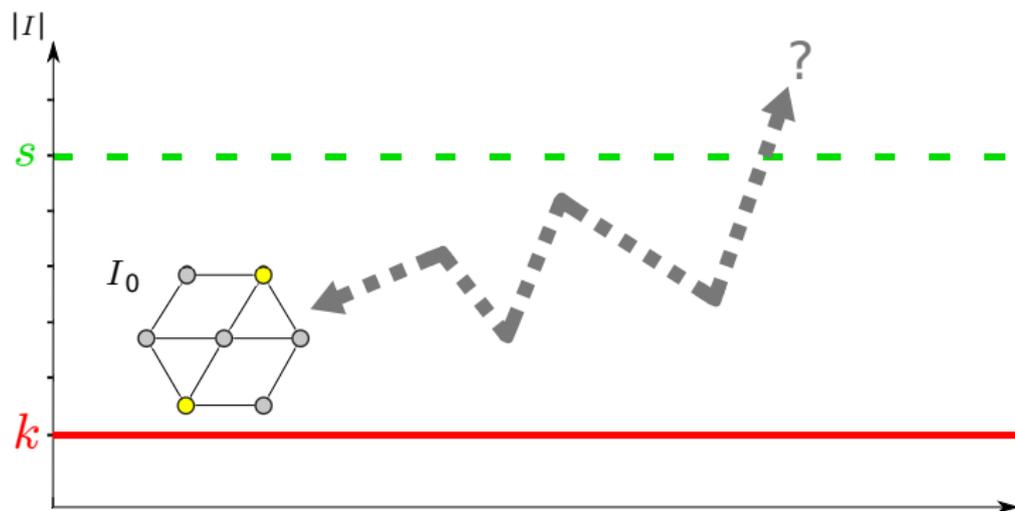
*Proof* : By adapting a result on **independent sets** and the **OPT-ISR** problem, analogous to OPT-DSR.

OPT-ISR deals with **independent set** reconfiguration.



## OPT-ISR

- **Input** : A graph  $G$ ,  $k, s \in \mathbb{N}$ , an independent set  $I_0$  with  $|I_0| \geq k$ .
- **Question** : Does  $G$  have an independent set  $I_s$  with  $|I_s| \geq s$  and  $I_0 \overset{\geq k}{\rightsquigarrow} I_s$  ?



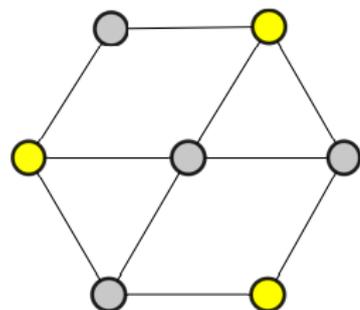
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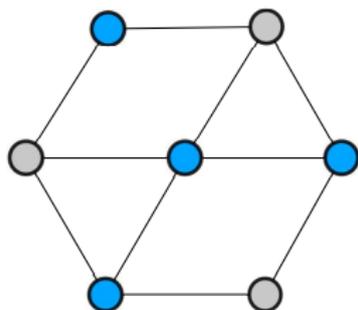
Theorem [Ito, Mizuta, Nishimura, Suzuki (2018)]

OPT-ISR is PSPACE-hard when the input graph has bounded pathwidth.

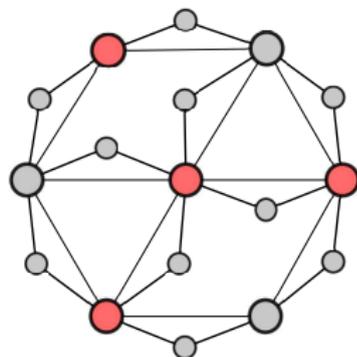
# Idea of the reduction



An independent set



A vertex cover  
(its complement)



An equivalent  
dominating set

## Corollary

**OPT-ISR** is PSPACE-hard  $\Rightarrow$  **OPT-VCR** is PSPACE-hard

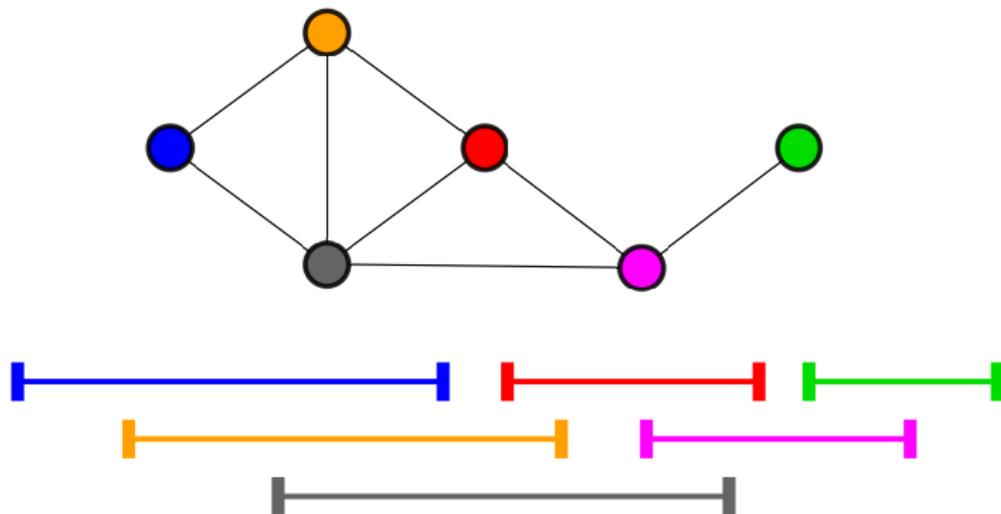
Theorem [B., Mizuta, Ouvrad, Suzuki (2020)]

**OPT-DSR** is PSPACE-hard.

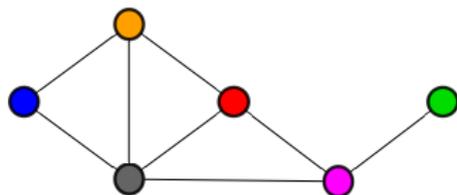
# Positive results

Theorem [B., Mizuta, Ouvrard, Suzuki (2020)]

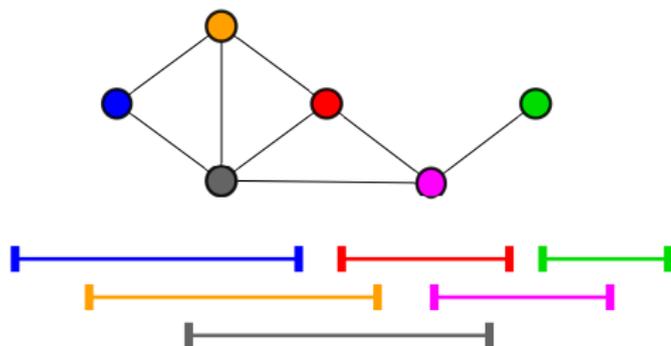
OPT-DSR can be solved in polynomial time on interval graphs.



*Proof :*  $(G, k, s, D)$ , with  $G$  an interval graph

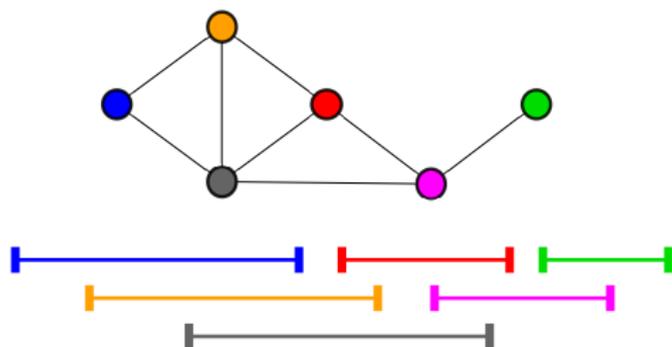


*Proof* :  $(G, k, s, D)$ , with  $G$  an interval graph



- We build in linear time in  $|G|$  a possible representation of  $G$  as a set of intervals.

*Proof* :  $(G, k, s, D)$ , with  $G$  an interval graph

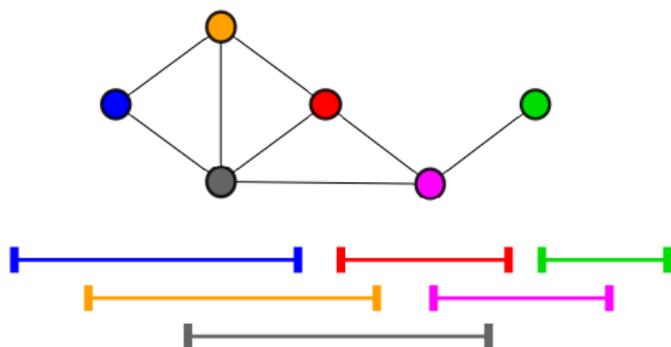


- We build a **minimum dominating set**  $D_m$  of  $G$ , in linear time in  $|G|$ .

**Lemma** [Haddadan *et al.* (2015)]

We can reconfigure  $D$  in  $D'$ , s.t.  $D_m \subseteq D'$ , under the bound  $|D| + 1$ , in linear time in  $|G|$ .

*Proof* :  $(G, k, s, D)$ , with  $G$  an interval graph

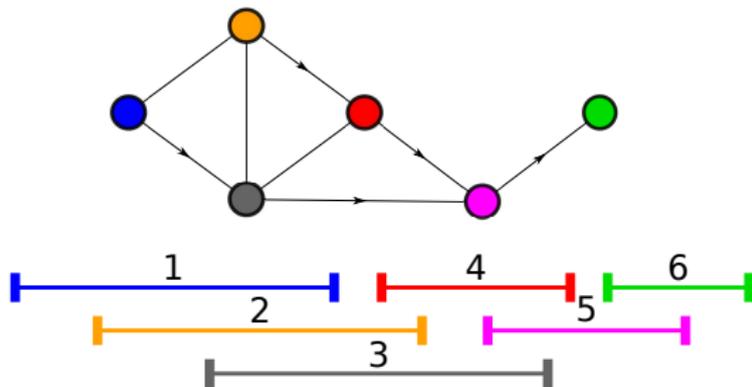


- We build a **minimum dominating set**  $D_m$  of  $G$ , in linear time in  $|G|$ .
- We thus have  $D \overset{k}{\leftrightarrow} D' \overset{k}{\leftrightarrow} D_m$ .  
We can answer yes if  $|D_m| \leq s$  and produce the corresponding sequence in linear time in  $|G|$ .

# Interval graphs

*Proof* : We build a minimum dominating set of  $G$ .

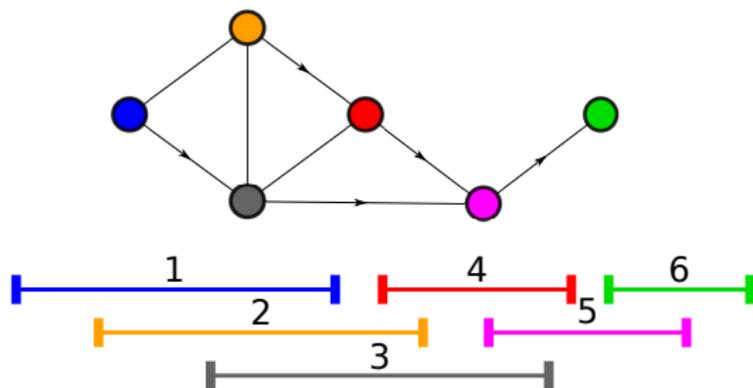
- Ordering : by ending time
- *favored neighbor* of  $v_i :=$  maximum neighbor of  $v_i$  in the ordering



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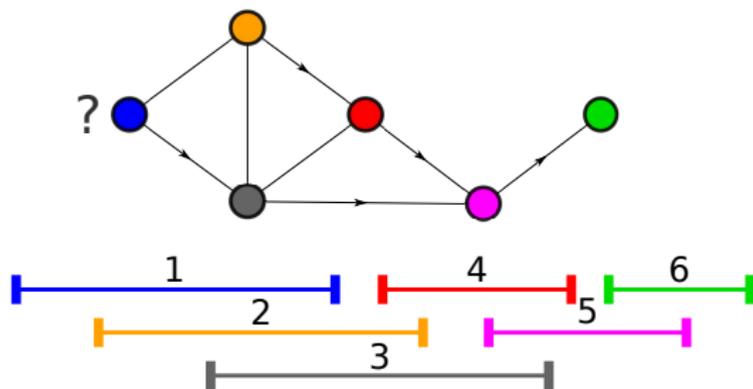


- **Algorithm** Traverse the vertices in order.
  - If  $v_i$  is dominated, skip it.
  - Otherwise, add its favored neighbor to the dominating set.

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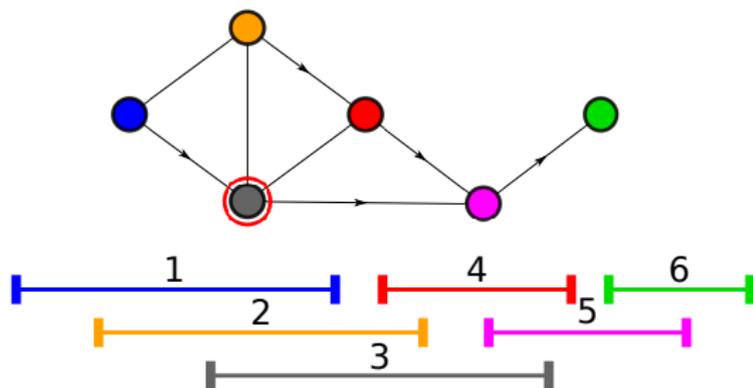


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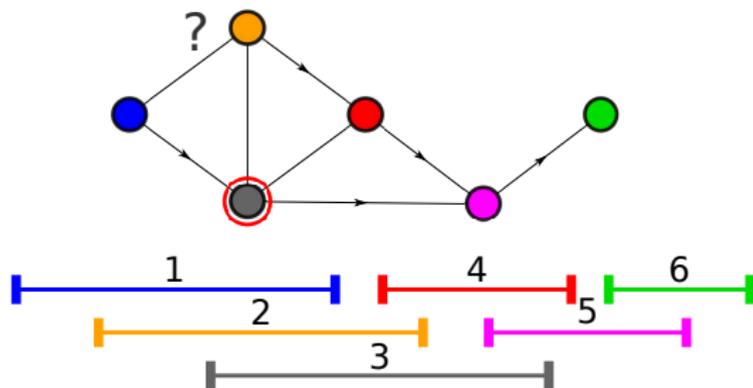


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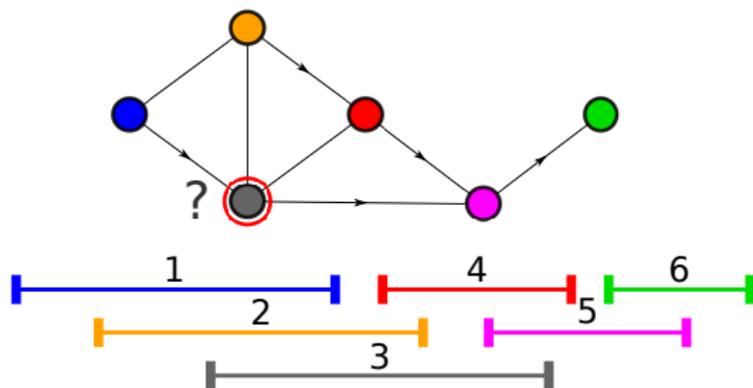


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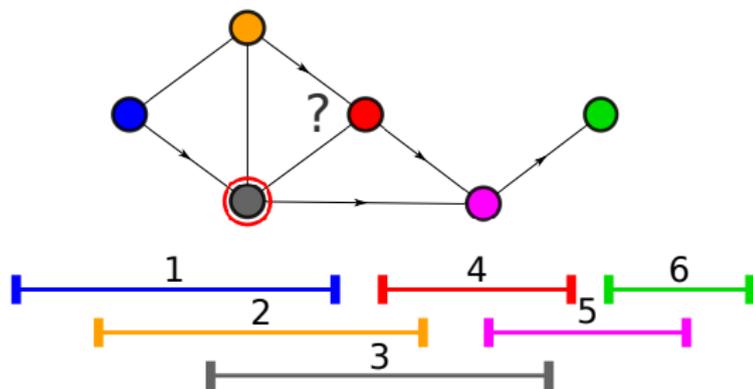


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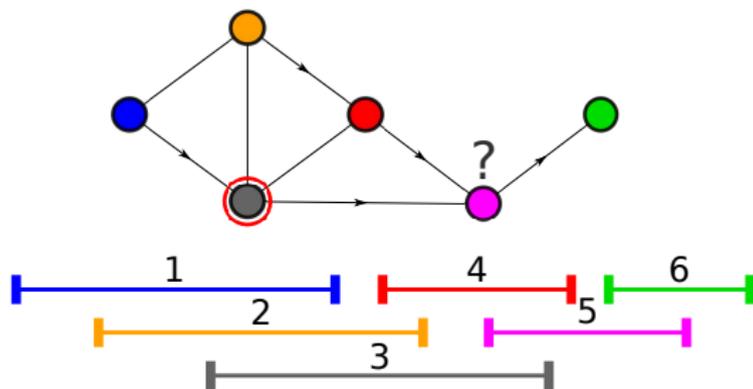


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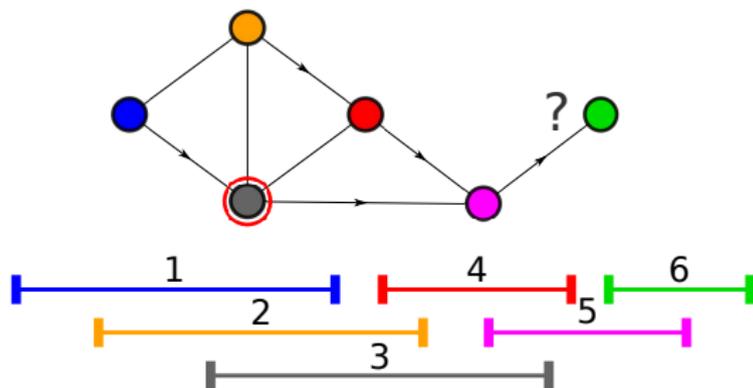


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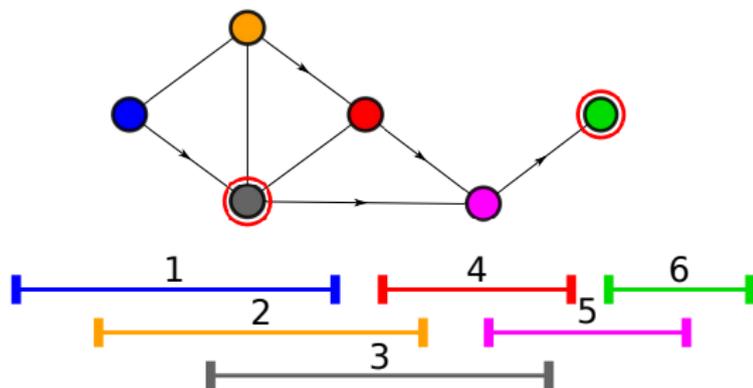


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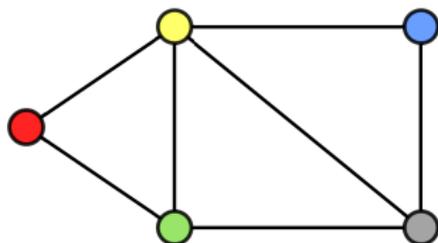
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# Parameterized complexity

A graph  $G$  is  $d$ -degenerate if it possesses a vertex  $v$  of degree  $\leq d$  and  $G - v$  is also  $d$ -degenerate.



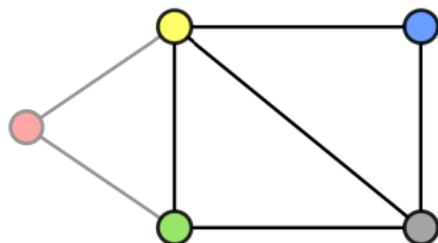
A 2-degenerate graph.

**Theorem [B., Mizuta, Ouvrard, Suzuki (2020)]**

OPT-DSR can be solved in time  $\text{FPT}(d + s)$ ,  
i.e. in time  $f(d + s) \times n^{O(1)}$  if  $|G| = n$ ;  
where  $d$  is the degeneracy of the graph  
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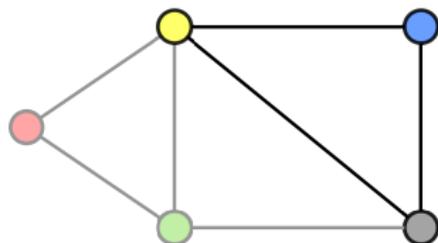
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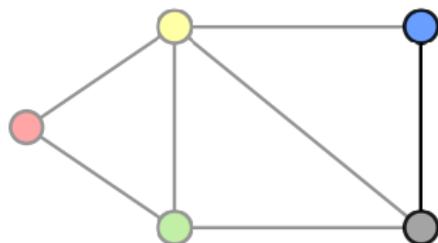
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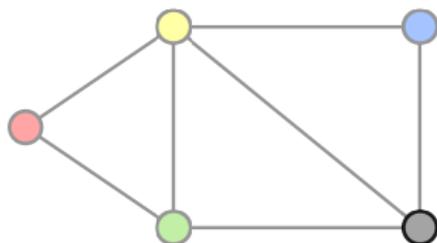
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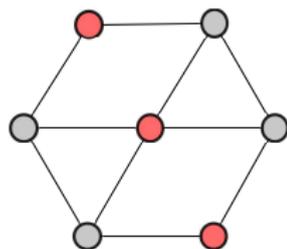
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*Proof :*

- **Trivial case** : If  $|D| = k$  and  $D$  is **minimal**, then  $D$  is *frozen*.  
We cannot remove nor add vertices,  
hence the instance  $(G, k, s, D)$  is positive iff  $|D| \leq s$ .  
→ We can test this condition in time  $O(|G|)$  ✓



Exemple pour  $k = 3$

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- If  $|D| = k$  and  $D$  is **not minimal**, then we can remove a vertex of  $D$  and reduce to the last case,  $|D| < k$ .

From now on we assume that  $|D| < k$ .

→ We can add at least 1 vertex to  $D$  without getting above  $k$

Theorem [B., Mizuta, Ouvrard, Suzuki (2020)]

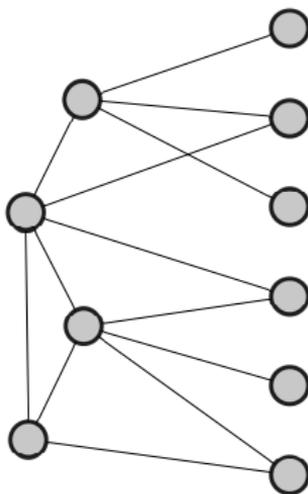
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- We compute  $\tau$  in time  $\text{FPT}(\tau)$ .
- 2 possibilities :
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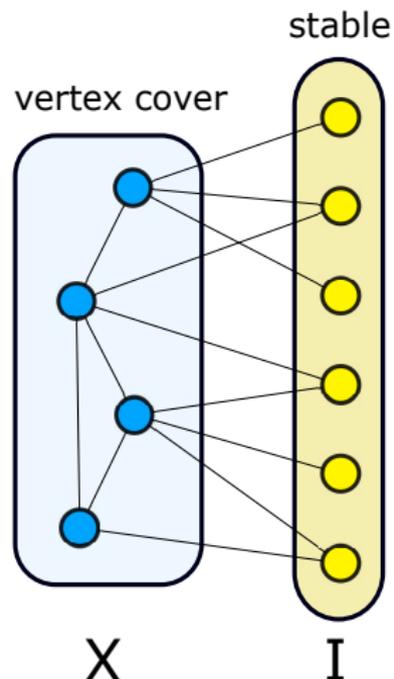
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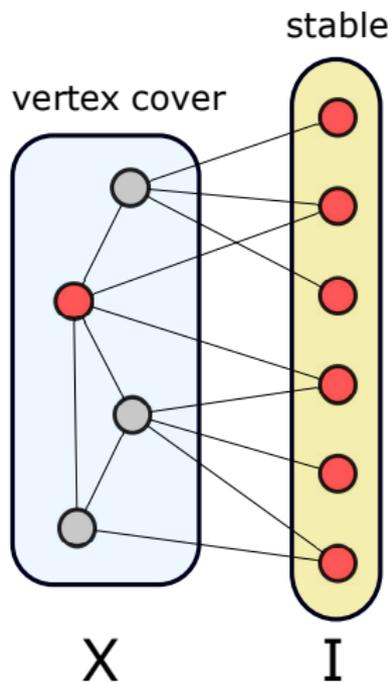
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→ we use the  $\text{FPT}(d + s)$ -time algorithm ✓
  - Or  $\tau \leq s$  : The instance is a **positive instance** in this case.  
To prove it, we will reconfigure  $D$  into a dominating set  $D'$   
that satisfies  $|D'| \leq \tau$ .



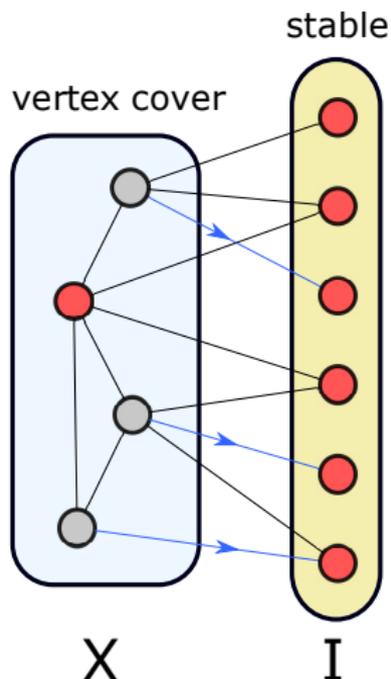
Let  $G$  be a graph.



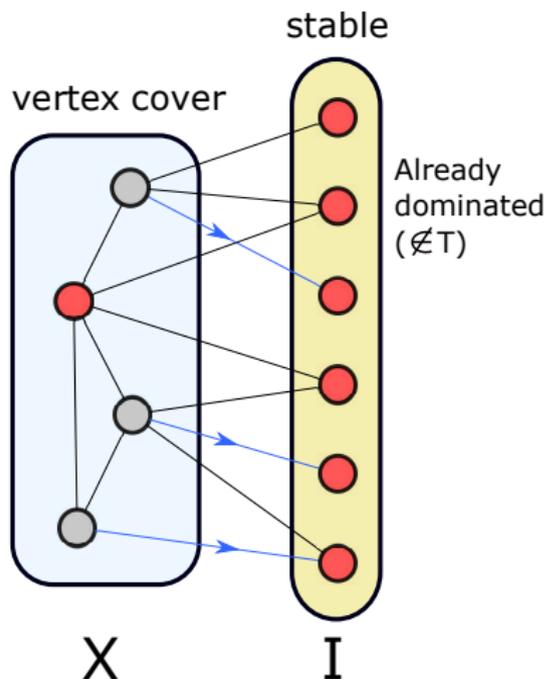
We build a minimum vertex cover in time  $\text{FPT}(\tau)$ .

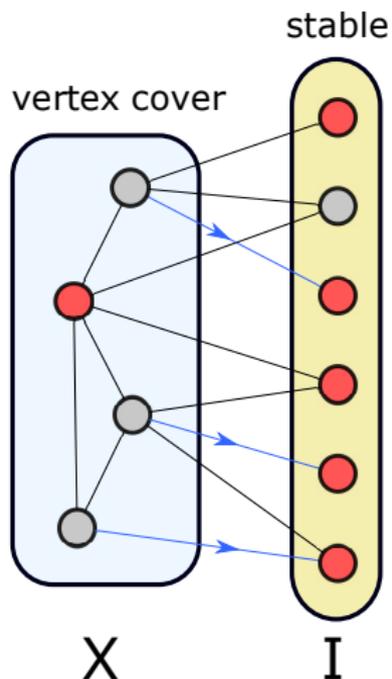


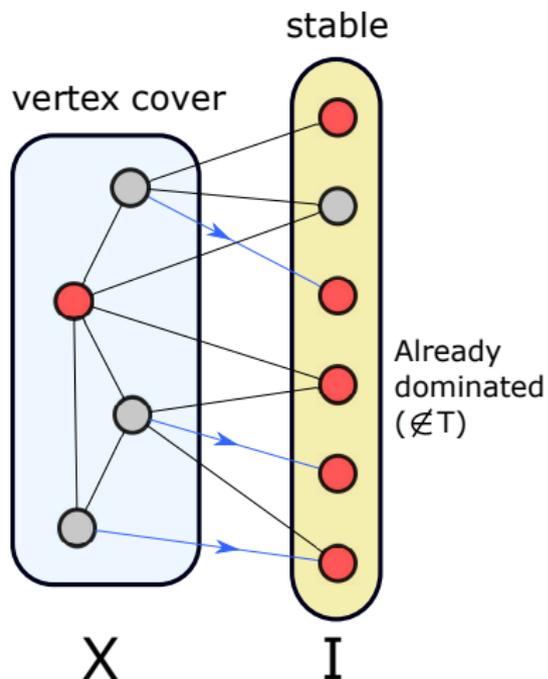
Let  $D$  be an initial dominating set.

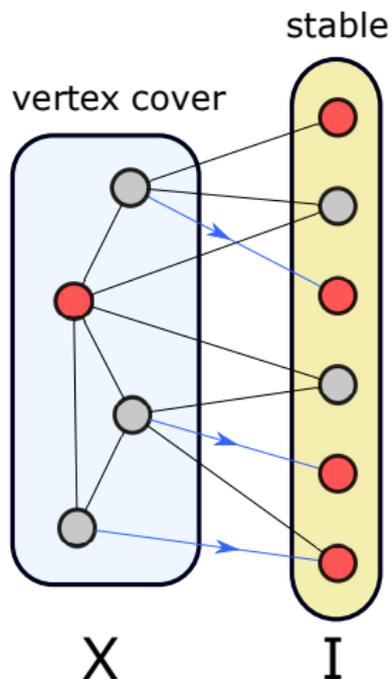


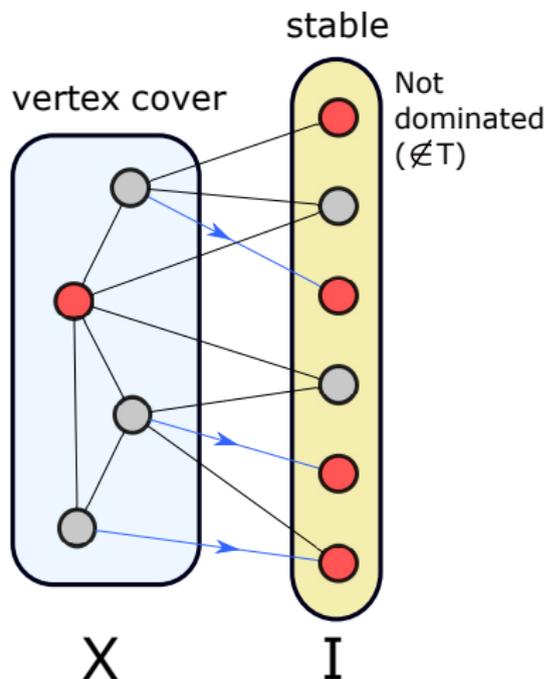
We associate to each  $v \in X \setminus D$  a neighbor  $t(v)$ .  
 Let  $T = \{t(v) \mid v \in X \setminus D\}$ .

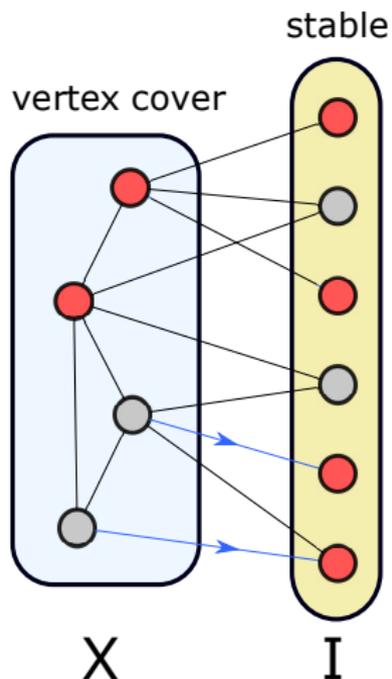


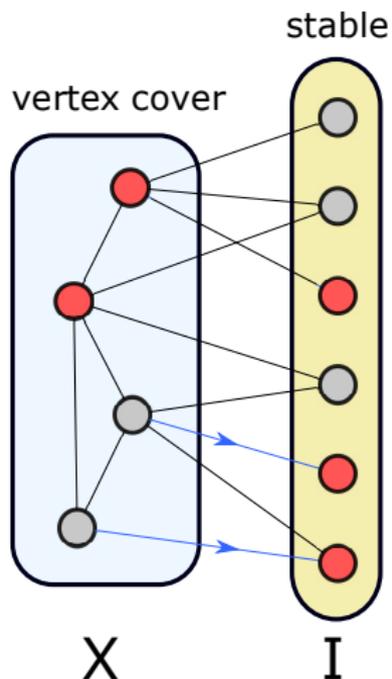


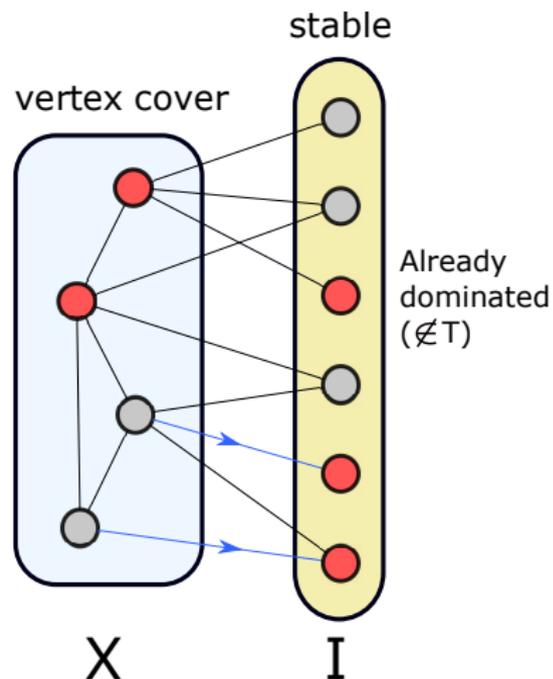


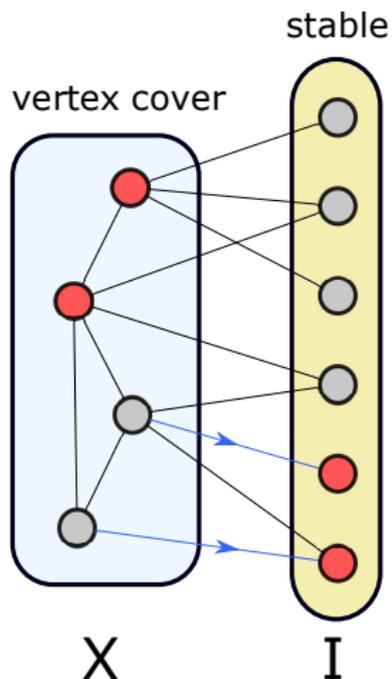


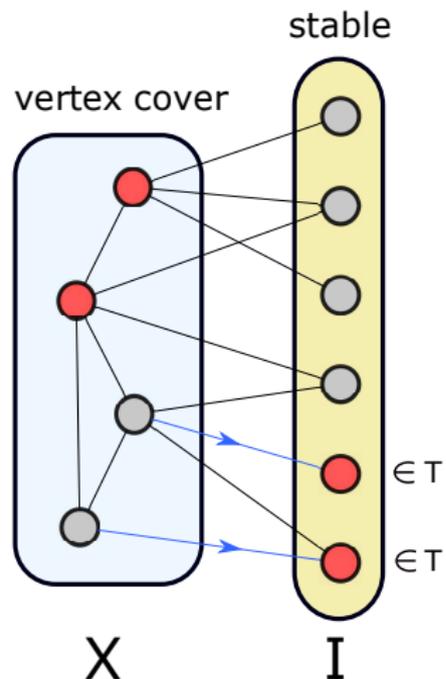






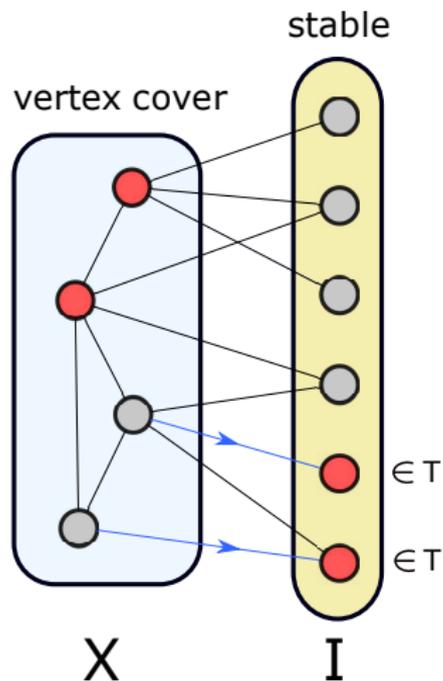






When  $(I \cap D') \subseteq J$ , we have :

$$\begin{aligned}
 |I \cap D'| &\leq |t^{-1}(I \cap D')| \\
 &\leq |X \setminus D'| \\
 &= \tau - |X \cap D'|
 \end{aligned}$$



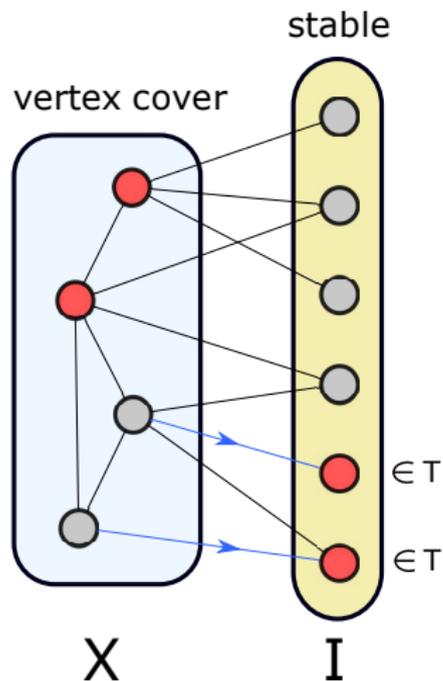
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$$\begin{aligned} |I \cap D'| &\leq |t^{-1}(I \cap D')| \\ &\leq |X \setminus D'| \\ &= \tau - |X \cap D'| \end{aligned}$$

Hence we have :

$$\begin{aligned} |D'| &= |I \cap D'| + |X \cap D'| \\ &\leq \tau \\ &\leq s \end{aligned}$$

Thus  $D'$  is a solution :  
the instance is positive.



## Complexity of OPT-DSR

- PSPACE-complete  
(even on bipartite, split and bounded pathwidth graphs)
- Polynomial on Interval graphs
- $\text{FPT}(d + s)$  ( $d = \text{degeneracy}$ ,  $s = \text{size of the solution}$ )
- $\text{FPT}(\tau)$  ( $\tau = \text{size of a minimum vertex cover}$ )

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**Thanks for your attention.**