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Strongly Stable and Maximum Weakly Stable Noncrossing Matchings

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Stable Marriage problem with Incomplete lists (SMI)

[Gale & Shapley 1962]

Input:

- men
- women
- each person's preference list

Example:

men: 1, 2, 3, 4, 5

women: a, b, c, d, e

1: a c b

2: c a

3: b a

4: c b d e

5: c d b

a: 2 1 3 4 5

b: 2 1

c: 1 2

d: 3 1 4

e: 4 3

Matching: a set of acceptable (**man**, **woman**)-pairs
(each person appears at most once)

Example: $\{(1,b), (2,c), (4,e)\}$

1:	a	<u>c</u>	(b)	a:	2	1	3	4	5
2:	(c)	a		b:	2	(1)	5		
3:	b	a		c:	<u>1</u>	(2)			
4:	c	b	<u>d</u>	(e)	d:	3	1	<u>4</u>	
5:	c	d	b		e:	(4)	3		

A **blocking pair** for a matching M (**informal**):

a pair of man and woman (m,w) such that

- (i) m and w are not matched together in M , but
- (ii) if (m,w) get matched, then both of m and w are improved.

A **blocking pair** is undesirable, because m and w have incentive to deviate from M to form a pair. As a result, M is broken.

Stable matching: a matching with no blocking pair

Example: $\{(1,a), (2,c), (4,d), (5,b)\}$

1: (a) c b	a: 2 (1) 3 4 5
2: (c) a	b: 2 1 (5)
3: b a	c: 1 (2)
4: c b (d) e	d: 3 1 (4)
5: c d (b)	e: 4 3

Known results [Gale & Shapley 1962]

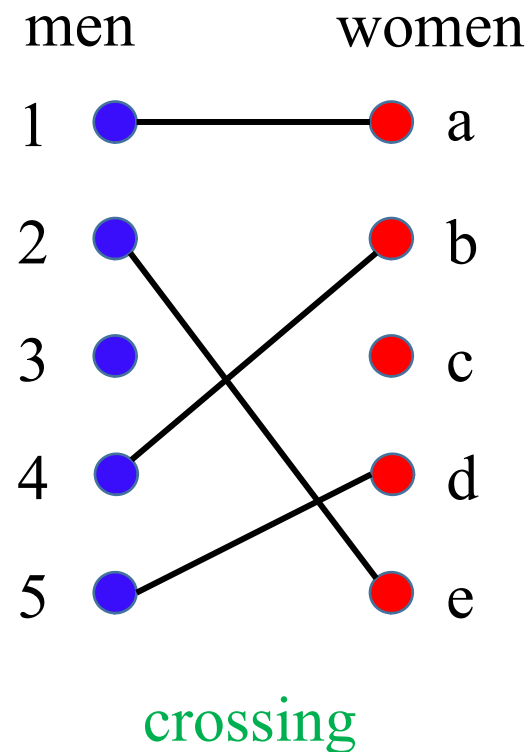
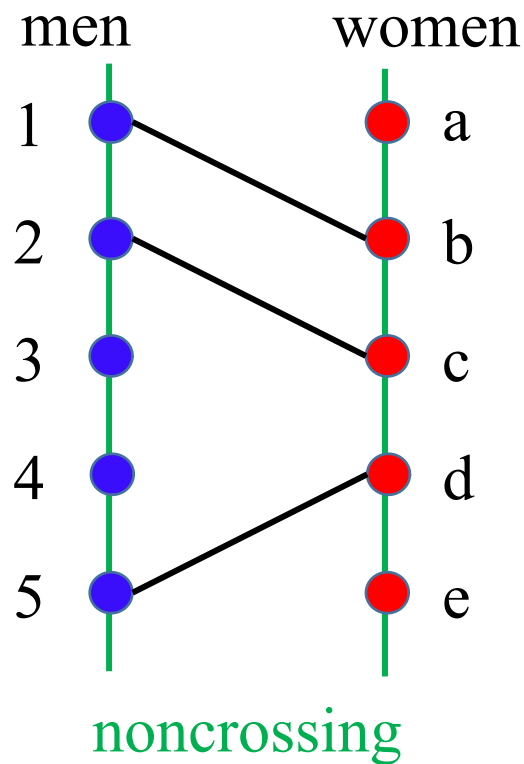
- For any instance, there exists at least one stable matching.
- There is a polynomial-time (-time) algorithm to find a stable matching. (**The Gale-Shapley algorithm**)

: the number of men (= the number of women)

Stable *Noncrossing* Matching [Ruangwises and Itoh, IWOCA 2019]

Men and women are aligned on two vertical parallel lines.

A matching is called **nocrossing** if no two edges cross each other, otherwise it is **crossing**.



The goal is to find a matching that is **stable** and **noncrossing**.

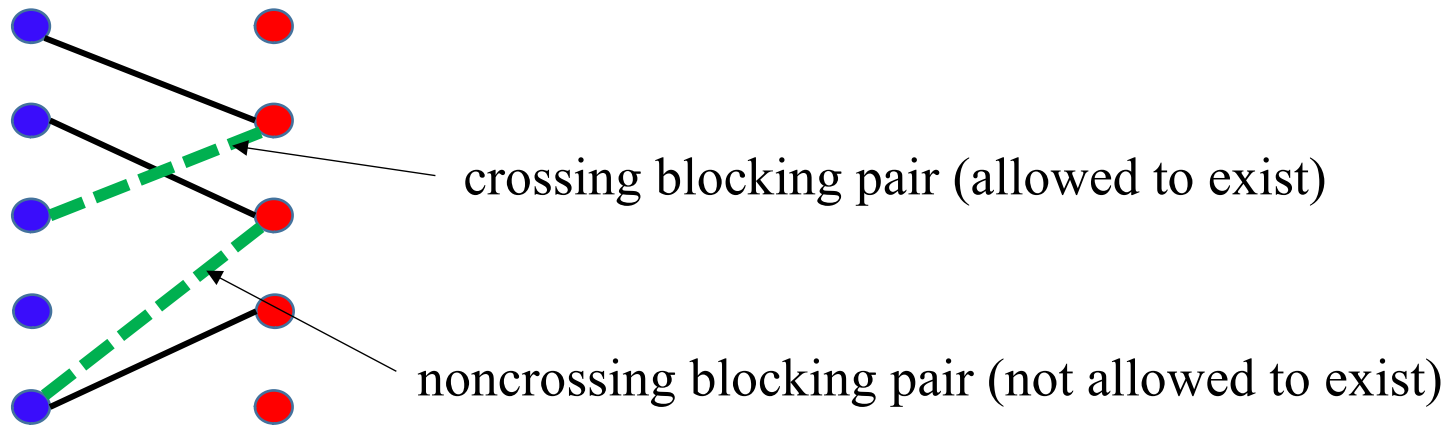
Two stability definitions [Ruangwises and Itoh, IWOCA 2019]

Strongly stable noncrossing matching (SSNM)

- The definition of stability is the same as the original definition.
- Exactly the intersection of stable matchings and noncrossing matchings.

Weakly stable noncrossing matching (WSNM)

The definition of stability is relaxed.



A matching is WSNM if

is noncrossing and admits no noncrossing blocking pair.

SSNM is always WSNM. (The converse is not true.)

Previous work [Ruangwises and Itoh, IWOCA 2019]

Results

- For any input, there exists a **WSNM**.
- A **WSNM** can be found in n^2 -time.
- There exists an instance that admits **WSNMs** of different sizes.
- There exists an instance that admits no **SSNM**.

Open questions

- What is the complexity of determining existence of an **SSNM**?
- What is the complexity of finding a maximum cardinality **WSNM**?

This work

- Both problems are solvable in polynomial time.
- Consider the problems where ties exist in preference lists (**SMTI**)

	Exist SSNM ?	Find WSNM	Find MAX WSNM
SMTI	1 $\binom{?}{2}$	[Ruangwises and Itoh]	2 $\binom{?}{4}$
SMTI-super	$\binom{2}{2}$		$\binom{4}{4}$
SMTI-strong	$\binom{3}{2}$		$\binom{4}{4}$
SMTI-weak	3 NP-complete		$\binom{4}{4}$

1. An $\mathcal{O}(n^2)$ -time algorithm for determining if SSNM exists (in SMI).

The rural hospitals theorem [Gale & Sotomayor 1985, Roth 1986]

In SMI, each person is either

- matched in all the stable matchings or
- unmatched in all the stable matchings.

We can identify all the matched persons in $\mathcal{O}(n^2)$ -time (by finding one stable matching using the Gale-Shapley algorithm).

(Men)

Matched: 1, 3, 4

Unmatched: 2, 5

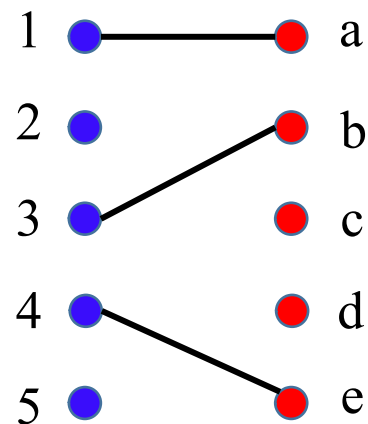
(Women)

Matched: a, b, e

Unmatched: c, d

There is only one way of matching 1, 3, 4 and a, b, e in a noncrossing manner.

→ Check if it is stable or not in $\mathcal{O}(n^2)$ -time.



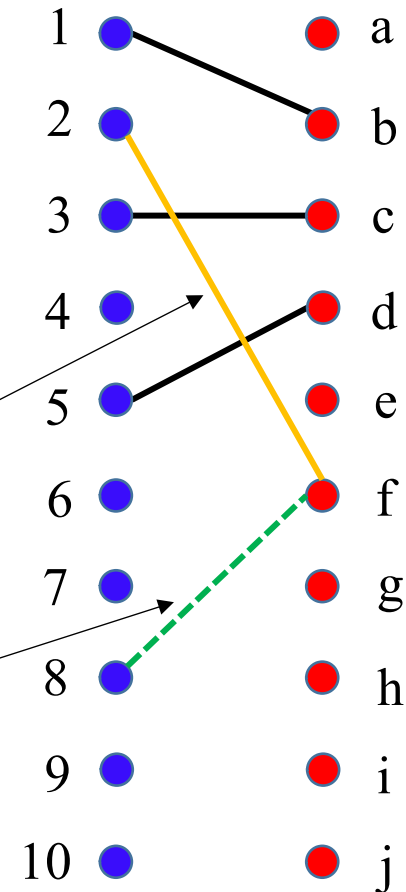
2. An $\mathcal{O}(n^2)$ -time algorithm for finding MAX WSNM (in SMI).

Basic strategy:

Add edges from top to bottom,
while **avoiding crossing** and
keeping stability.

We don't have to care whether (2,f) is a blocking pair or not.
(because even if it is a blocking pair, it's a **crossing blocking pair**).

Can we add this edge?



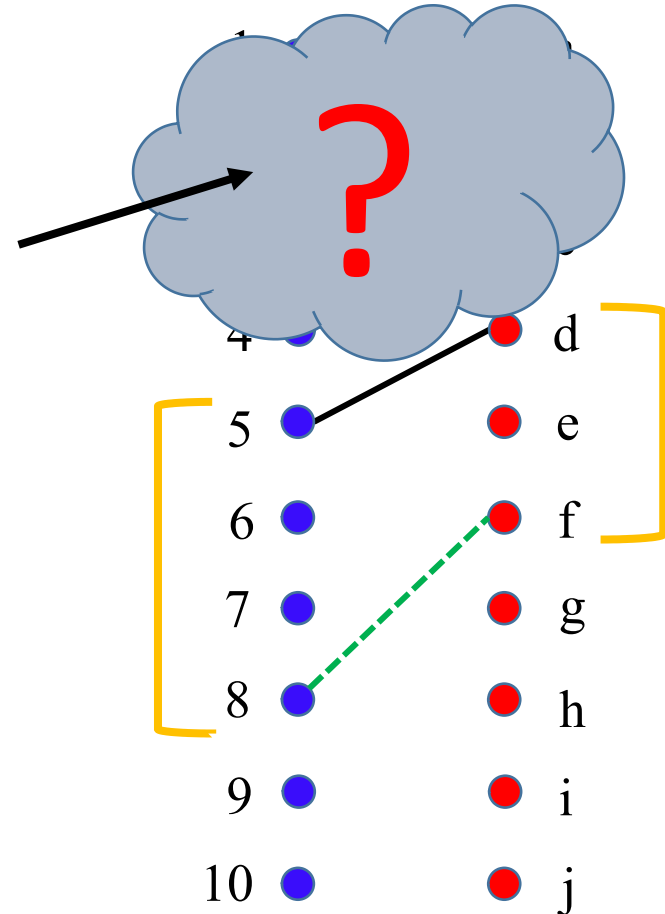
Noncrossing blocking pairs can arise only between 5, 6, 7, 8 and d, e, f.

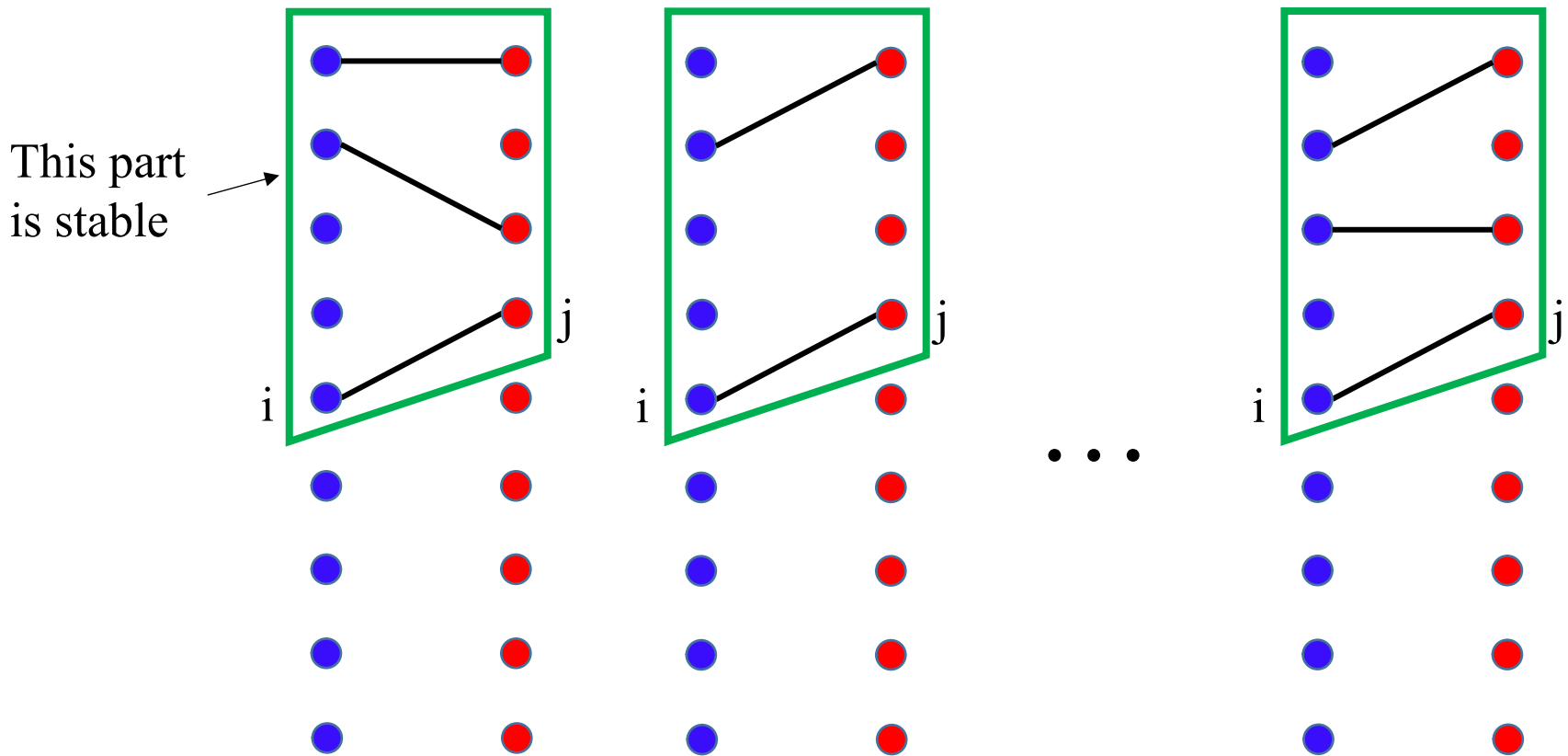
We don't need to know this part.

Only the important fact is that (5, d) is the **frontier edge**.



Dynamic programming



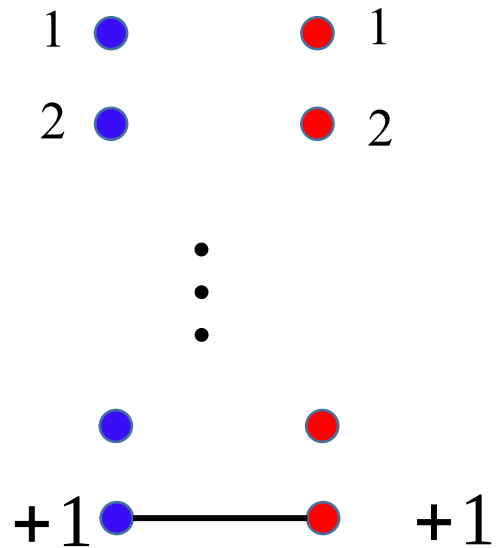


: the maximum size of noncrossing matchings such that

- the frontier edge is i and j
- matching is stable up to i .

(If no such matching exists, 0 .)

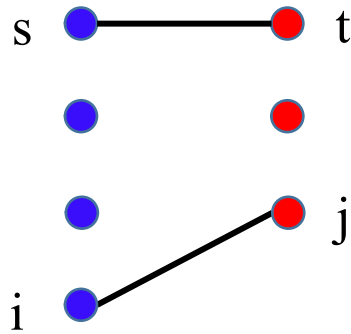
We add a dummy man at the bottom who are matched in any stable matching.



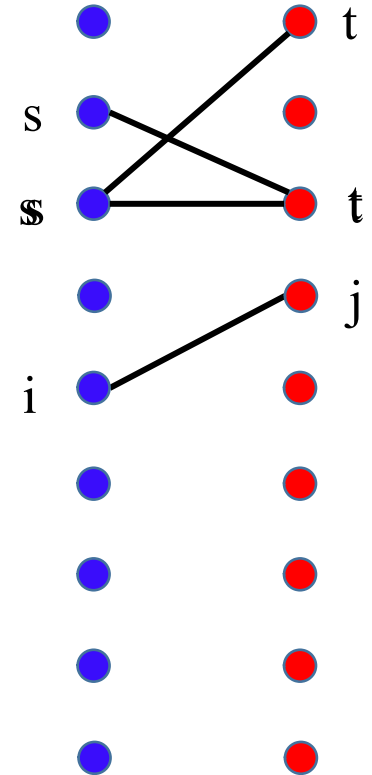
The answer to the original problem is .

Definition

are **nonconflicting**
if there is no blocking pair
between .



can be consecutive edges in a
WSNM.



Computing

are nonconflicting}

Time-complexity

are nonconflicting}

- For computing one s , there are at most n candidates for s .
- For a fixed pair of s and t , checking if they are conflicting can be done in $O(1)$ -time.
 - One s can be computed in $O(n)$ -time.
- There are at most n s .
 - The total time-complexity is $O(n^2)$.



Can be reduced to $O(n)$ by devising efficient data structures.

3. NP-completeness of determining if SSNM exists (in SMTI-weak).

Definition of SMTI-weak

- Preference lists may contain ties.
- (i, j) is a blocking pair if both i and j are **strictly** improved.

1: <u>a</u> c (b)	a: (2 <u>1</u>) (3 4) 5
2: (c) a	b: (2 (1))
3: <u>d</u> (a)	c: 1 5 (2)
4: (c b) (d) e	d: (<u>3</u> 1 (4))
5: d (b c)	e: 4 3

→ (1, a) is a blocking pair

→ (3, d) is **not** a blocking pair

In this stability definition, we consider existence of a matching that is **stable** & **noncrossing**.

NP-hardness

Reduction from 3SAT [Cook 1971]

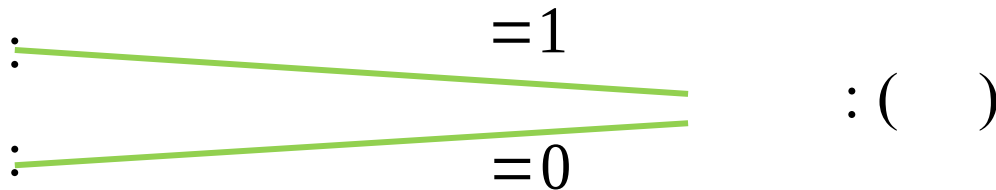
(Without loss of generality, we may assume that each clause contains **exactly** 3 literals.)

A variable gadget

For a variable

Men and , woman

Preference lists

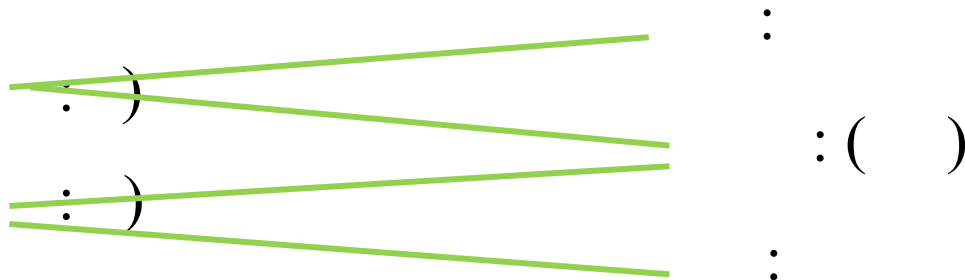


A clause gadget

For a clause

Men , , women , ,

Preference lists



is satisfied by the ~~2nd~~ literal.

(because the 3rd woman is single)

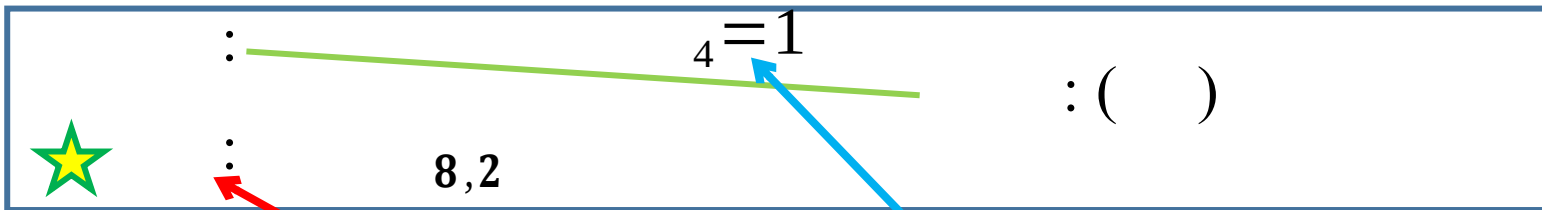
Connecting variable gadgets and clause gadgets

Do the following for each literal.

As an example, consider the 2nd literal of .
 Suppose that it is .

$$8 = (\bigcirc \vee \neg \quad 4 \vee \bigcirc)$$

Variable gadget for

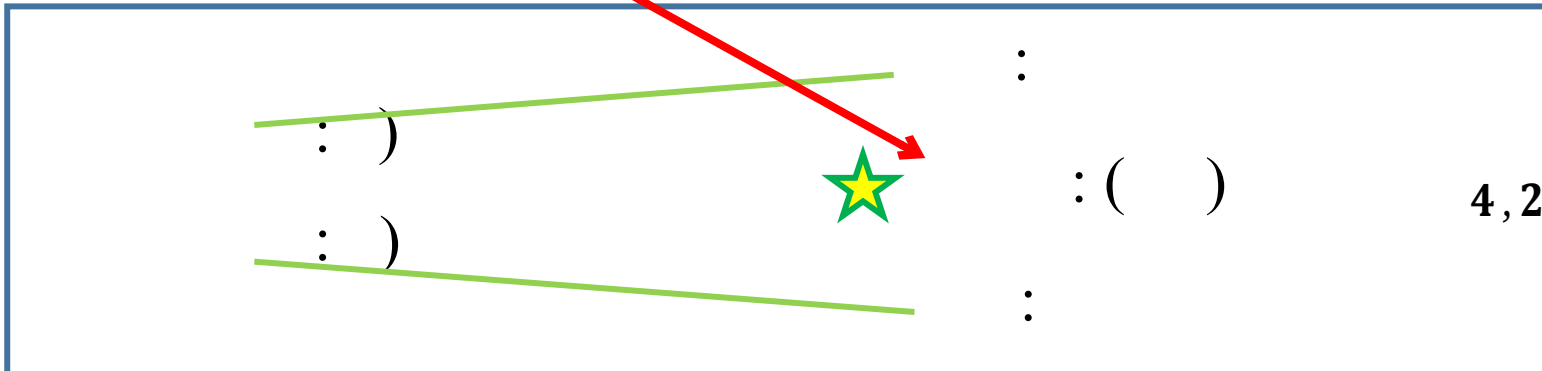


blocking pair

contradiction

Clause gadget for

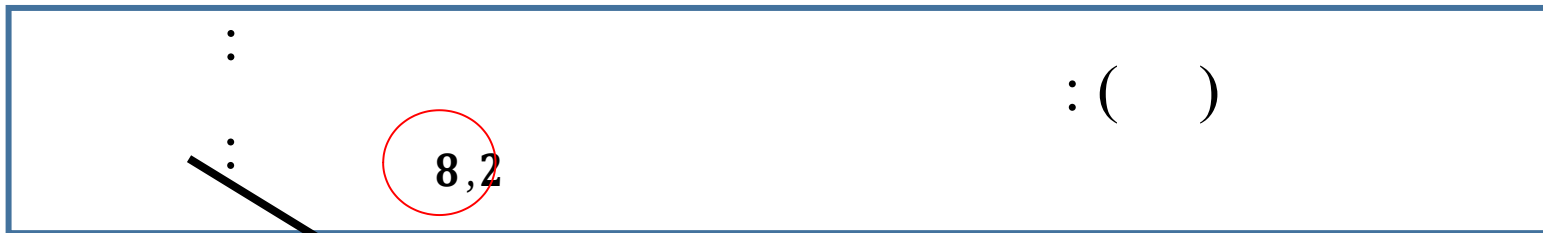
is satisfied by the 2nd literal.



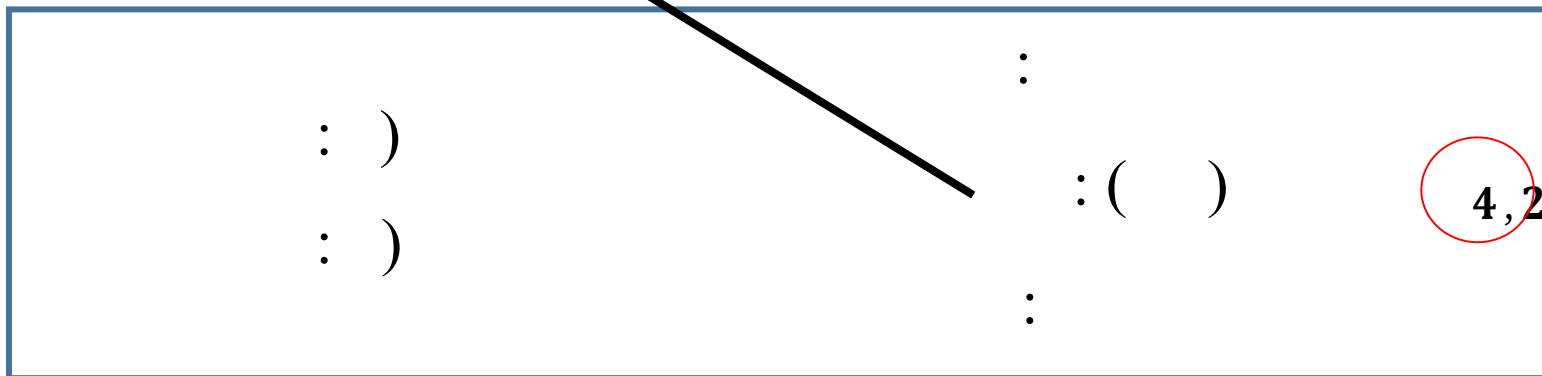
For this idea to work, we have to rule out **illegal match**.

a man in a variable gadget — a woman in a clause gadget

variable gadget for



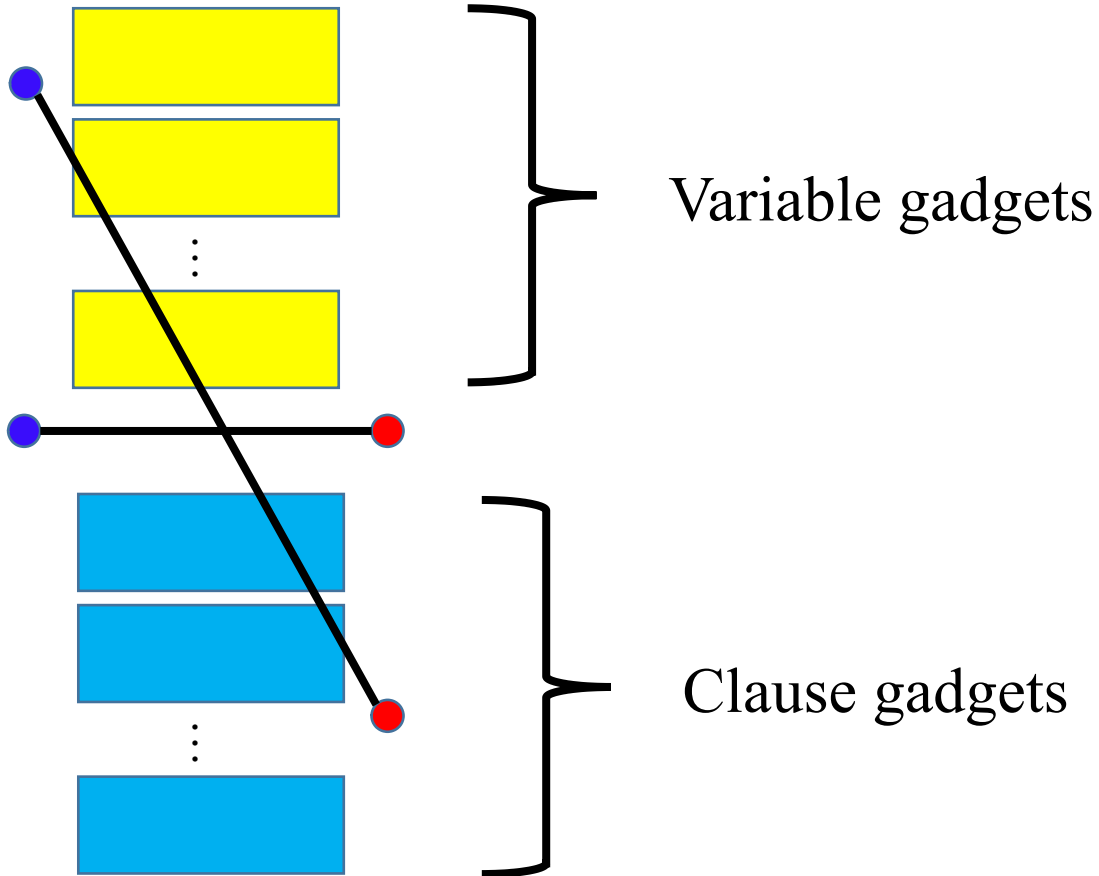
clause gadget for



A separator



Always be matched.



Conclusion

Summary

	Exist SSNM ?	Find WSNM	Find MAX WSNM
SMI	$\binom{2}{2}$	[Ruangwises and Itoh]	} $\binom{4}{4}$
SMTI-super	$\binom{2}{2}$		
SMTI-strong	$\binom{3}{3}$		
SMTI-weak	NP-complete		

Future work

- Reducing .
- Complexity of optimization variant.
(noncrossing \rightarrow minimum number of crossings.)
- Placing agents in other formations (rather than two parallel lines).