

On Proper Labellings of Graphs with Minimum Label Sum

Julien Bensmail Foivos Fioravantes Nicolas Nisse

Université Côte d'Azur, CNRS, Inria, I3S, France

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COATI



Introduction

Proper k -labellings

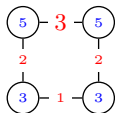
Given graph $G = (V, E)$, define:

- **k -labelling** of G : function $\ell : E \mapsto \{1, \dots, k\}$.
- The **colouring** of G **induced by** ℓ : c_ℓ s.t. for each $v \in V$,
 $c_\ell(v) = \sum_{u \in N(v)} \ell(uv)$.

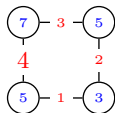
Proper labelling

The labelling ℓ is a **proper k -labelling** of G if c_ℓ is a proper vertex-colouring.

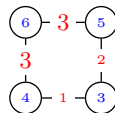
Examples:



3-labelling



proper 4-labelling



proper 3-labelling

Previous work

Let $G = (V, E)$ be a nice (no component isomorphic to K_2) graph.
Define $\chi_\Sigma(G)$: minimum k s.t. G admits a proper k -labelling.

1-2-3 Conjecture: $\chi_\Sigma(G) \leq 3$ (2004, Karoński, Łuczak, Thomason)

- Verified for:
 - Complete graphs
 - Graphs with chromatic number ≤ 3
- In general: $\chi_\Sigma(G) \leq 5$ (2010, Kalkowski, Karoński, Pfender)
- G is regular $\Rightarrow \chi_\Sigma(G) \leq 4$ (2019, Przybyło)

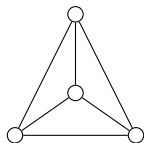
Complexity

- Deciding if $\chi_\Sigma(G) = 2$ is \mathcal{NP} -Complete (2011, Dudek, Wajc)
- G is bipartite \Rightarrow deciding if $\chi_\Sigma(G) = 2$ is easy (2016, Thomassen, Wu, Zhang)

Equivalent definition

Turning a graph into a **locally irregular** (adjacent vertices have different degree) multigraph by multiplying edges.

Inspired by (1988, Chartrand, Erdős, Oellermann)

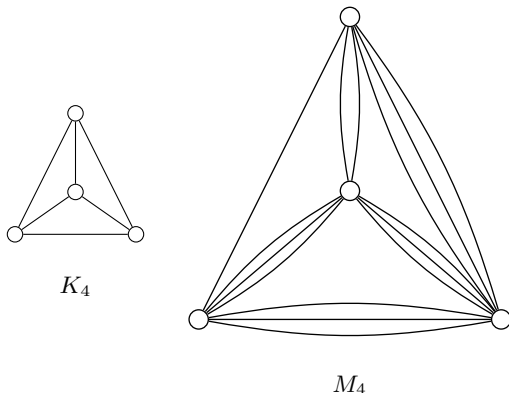


K_4

Equivalent definition

Turning a graph into a **locally irregular** (adjacent vertices have different degree) multigraph by multiplying edges.

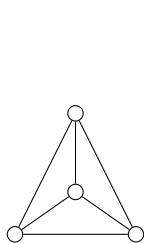
Inspired by (1988, Chartrand, Erdős, Oellermann)



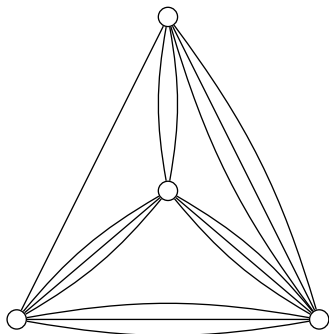
Equivalent definition

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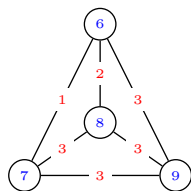
Inspired by (1988, Chartrand, Erdős, Oellermann)



K_4



M_4



Proper labelling of K_4

Interesting to minimise the edges of the multigraph \rightarrow the sum of the labels.

Other objectives

1-2-3-Conjecture: minimisation of max used label.

Different objectives:

- Minimise number of distinct colours (2019, Baudon, Bensmail, Hocquard, Senhaji, Sopena)
- Minimise the maximum colour (2019, Bensmail, Li, Li, Nisse)
- **Minimise sum of labels used (our work)**

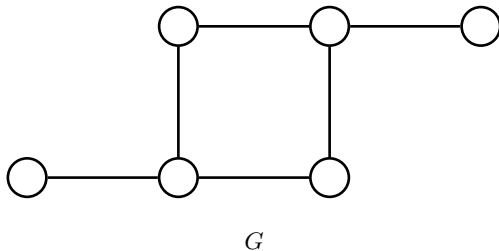
Notation:

Let $G = (V, E)$ be a nice (no component isomorphic to K_2) graph.

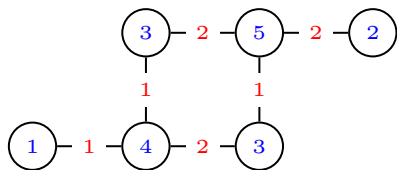
- $\sigma(\ell) = \sum_{e \in E} \ell(e)$, where ℓ is a proper k -labelling of G
- $mE_k(G) = \min\{\sigma(\ell) : \ell \text{ is a proper } k\text{-labelling of } G\}$
- $mE(G) = \min\{mE_k(G) : k \geq \chi_\Sigma(G)\}$

Warm-up

Example 1



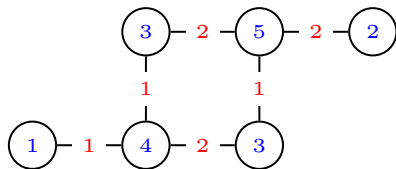
Example 1



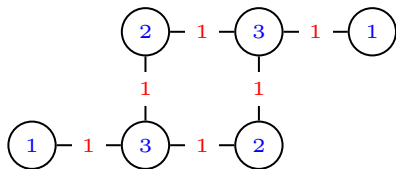
$$G, \sigma(\ell) = 9$$

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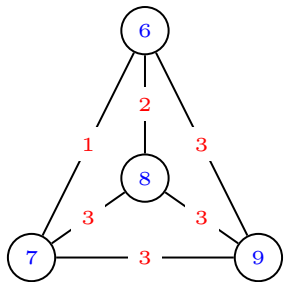
$$mE(G) = 6$$

Lemma

$mE(G) = |E| \Leftrightarrow G$ is locally irregular.

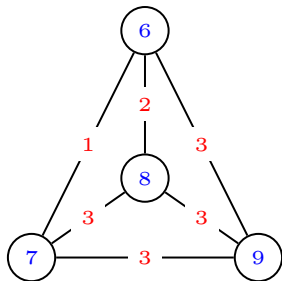
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Example 2

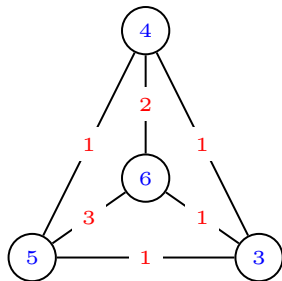


$$K_4, \sigma(\ell) = 15$$

Example 2



$$K_4, \sigma(\ell) = 15$$



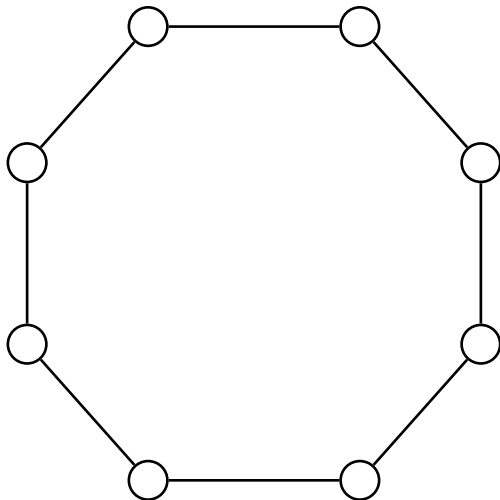
$$mE(K_4) = 9$$

Theorem

Let $K_n = (V, E)$ be the complete graph on $n \geq 3$ vertices. Then:

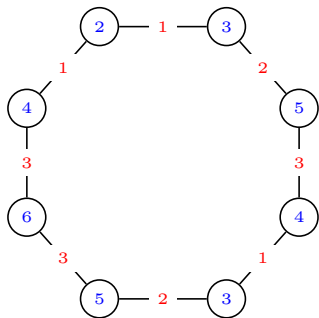
- if $n = 3$, $mE(K_3) = 2|E|$
- if $n \equiv 0 \pmod{4}$ or $n \equiv 1 \pmod{4}$, $mE(K_n) = \frac{3}{2}|E|$
- otherwise, $mE(K_n) = \lceil \frac{3}{2}|E| \rceil$

Example 3



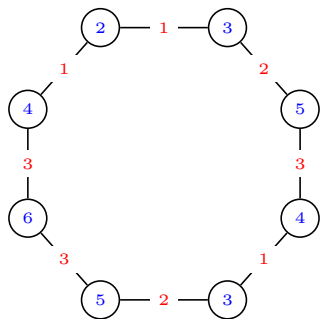
C_8

Example 3

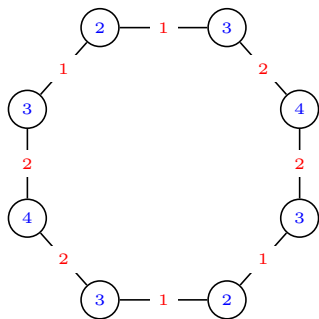


$$C_8, \sigma(l) = 16$$

Example 3



$$C_8, \sigma(\ell) = 16$$



$$mE(C_8) = 12$$

Theorem

Let $C_n = (V, E)$ be the cycle of length n . Then:

- if $n \equiv 0 \pmod{4}$, $mE(C_n) = mE_2(C_n) = \frac{3}{2}|E|$
- if $n \equiv 2 \pmod{4}$, $mE(C_n) = mE_3(C_n) = \frac{3}{2}|E| + 3$
- otherwise, $mE(C_n) = mE_3(C_n) = \lceil \frac{3}{2}|E| \rceil + 1$

Our results

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Tight results for:

- $mE(K_n) = \frac{3}{2}|E| + c$
- $mE(K_{n,n}) = |E| + \sqrt{|E|}$
- $mE(C_n) = \frac{3}{2}|E| + c$

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Peculiar behaviour

Larger labels may help

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Peculiar behaviour

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Complexity

- For $k \in \mathbb{N}$, finding $mE_k(G)$ is \mathcal{NP} -hard, G : planar bipartite graph
- Polynomial algorithm, k fixed, G : bounded treewidth

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Conjecture: for every graph $G = (V, E)$, $mE(G) \leq 2|E|$

- Graphs with particular chromatic numbers
- Trees

Our results

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Using larger labels

Larger labels may help

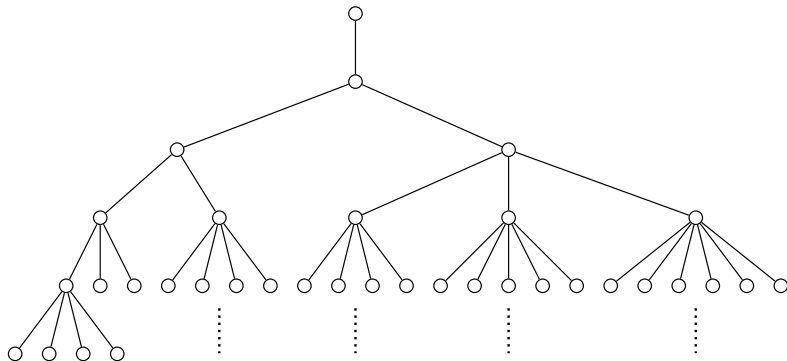
Theorem

For any $k \geq 2$, there exists a tree T_k s.t. $mE_{k+1}(T_k) < mE_k(T_k)$.

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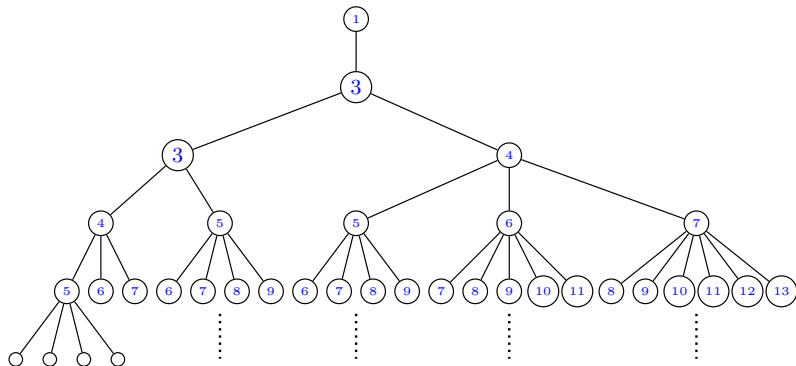


T_2

Larger labels may help

Theorem

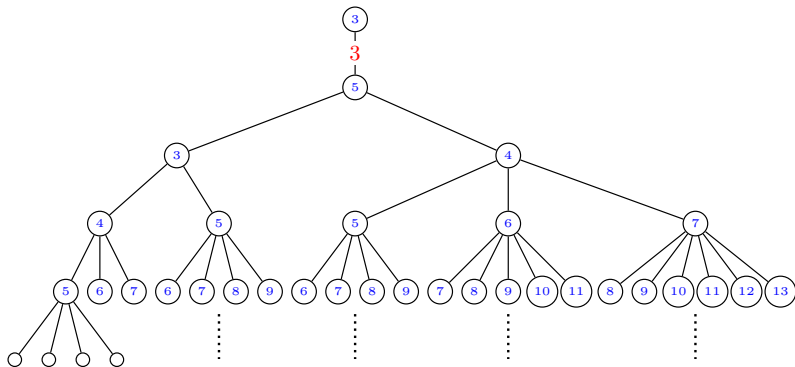
For any $k \geq 2$, there exists a tree T_k s.t. $mE_{k+1}(T_k) < mE_k(T_k)$.



T_2 , all labelled 1

Larger labels may help

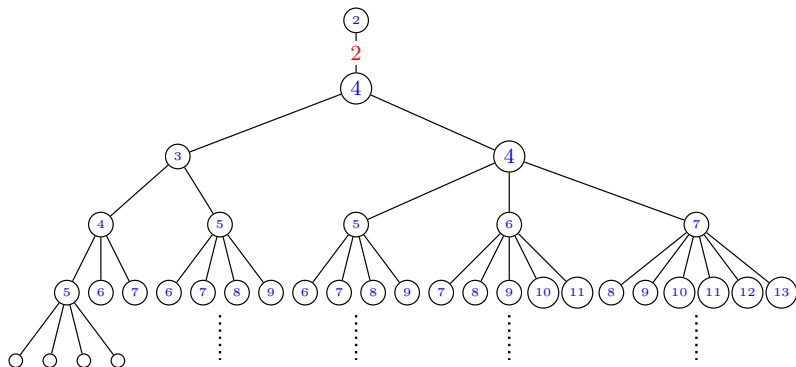
T_2 such that $mE_3(T_2) < mE_2(T_2)$



- $mE_3(T_2) = |E(T_2)| + 2 \Rightarrow$ better than any 2-labelling that uses label 2 more than two times.

Larger labels may help

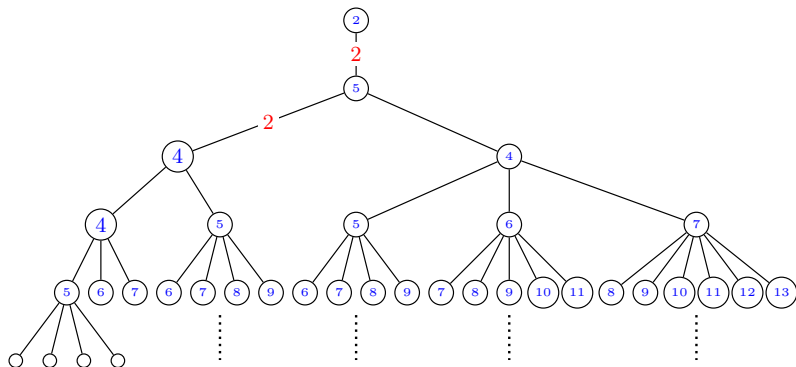
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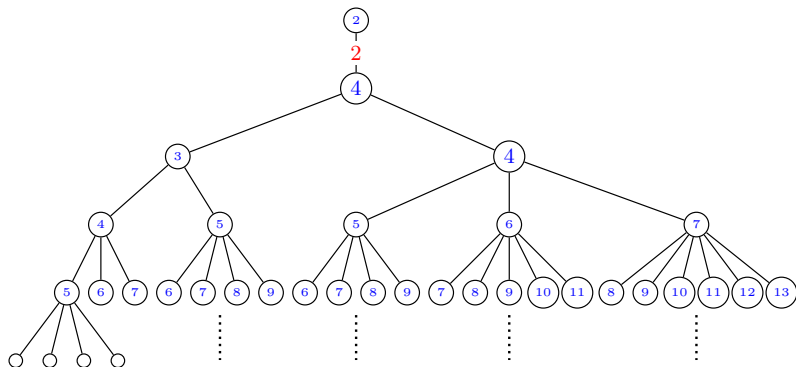
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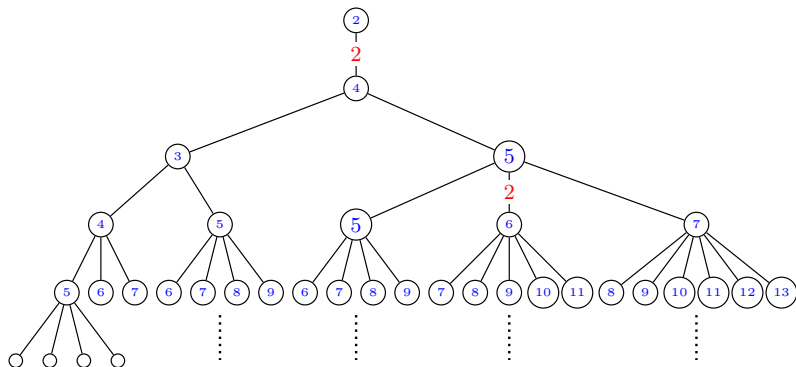
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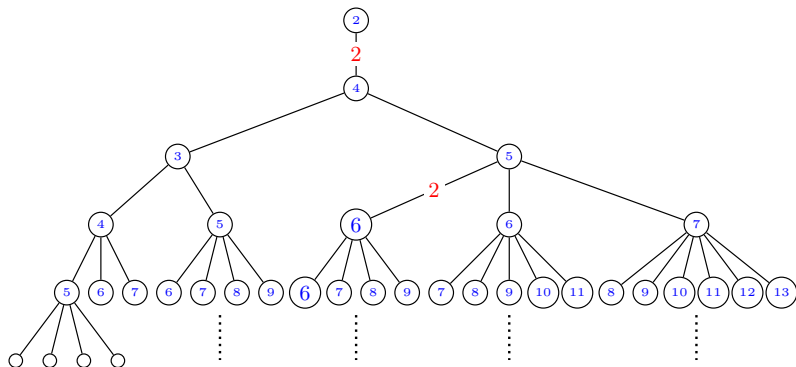
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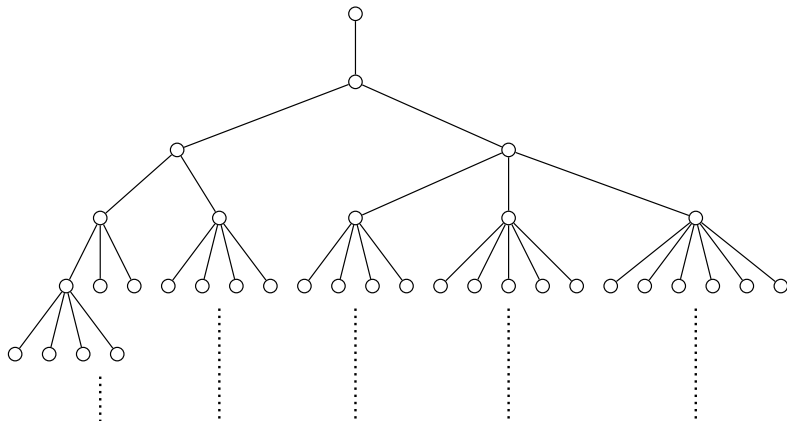


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Theorem

For any $k \geq 2$, there exists a tree T_k s.t. $mE_{k+1}(T_k) < mE_k(T_k)$.



$k + 1$ can be arbitrarily better than k

Complexity

Complexity

Theorem

Let G be a planar bipartite graph, $k \geq 2$ and $q \in \mathbb{N}$. Deciding if $mE_k(G) \leq q$ is \mathcal{NP} -Complete.

Reduction from PLANAR MONOTONE 1-IN-3 SAT.

PLANAR MONOTONE 1-IN-3 SAT^(2006, Mulzer, Rote)

Input: 3CNF formula \mathcal{F} .

Question: exist a 1-in-3 truth assignment satisfying \mathcal{F} ?

Planar: Corresponding graph is planar

Monotone: Only positive variables

1-in-3: Only one *True* variable per clause

Calculating $mE_2(G)$ is \mathcal{NP} -Complete

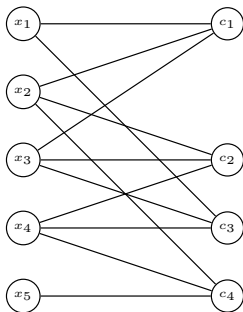
$$\mathcal{F} = (x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_3 \vee x_4) \wedge (x_2 \vee x_4 \vee x_5).$$

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Planar bipartite graph $G = (V, E)$ s.t.:

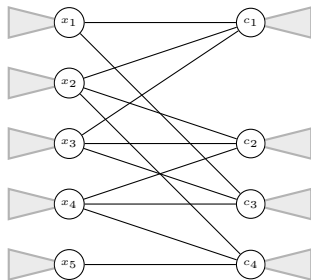
\mathcal{F} is 1-in-3 satisfied $\Leftrightarrow mE_2(G) = |E| + c$ ($c = |\text{clauses of } \mathcal{F}|$ (here $c = 4$)).



G' corresponding to \mathcal{F}

Calculating $mE_2(G)$ is \mathcal{NP} -Complete

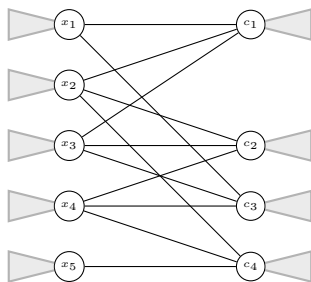
$G' \rightarrow$ add gadgets $\rightarrow G$.



Planar bipartite graph G

Calculating $mE_2(G)$ is \mathcal{NP} -Complete

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Planar bipartite graph G

Gadgets:

- Trees
- Non gadget edges adjacent to x_i : all 1 or all 2
- Non gadget edges adjacent to c_i : exactly one 2

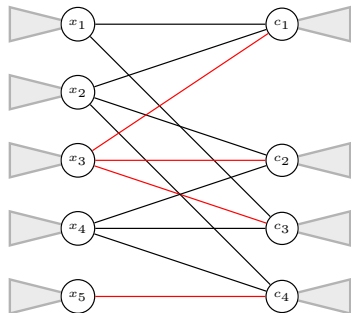
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Planar bipartite graph $G = (V, E)$ s.t.:

\mathcal{F} is 1-in-3 satisfied $\Leftrightarrow mE_2(G) = |E| + 4$

x_i *True* in $\mathcal{F} \Leftrightarrow$ all edges of G' adjacent to x_i labelled **2**.



Red edges labelled 2

Conclusion

Open questions

- For $k \in \mathbb{N}$, determining $mE_k(G)$ is \mathcal{NP} -Complete. But what about $mE(G)$? No general bound for k , even for trees.
- Adapt algorithm to compute $mE(G)$, for G of bounded treewidth.

Conjecture

For every graph $G = (V, E)$, $mE(G) \leq 2|E|$.

Conjecture

For every bipartite graph $G = (V, E)$, $mE(G) \leq \frac{3}{2}|E| + c$.

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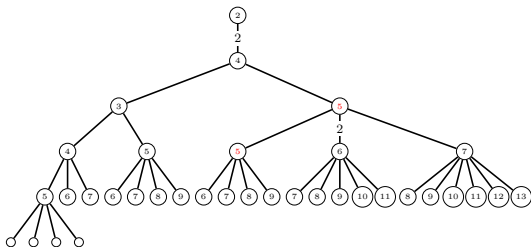
Thank you

Extra

Larger labels

For every $k \geq 2$, $T_k = (V, E)$ is such that:

- ① $mE_{k+1}(T_k) = |E| + k$
- ② By any proper k -labelling of T_k , every edge that is labelled $\neq 1$, must have at least one descendant¹ labelled $\neq 1$ (if such an edge exists)
- ③ $mE_k(T_k) \propto \text{height of } T_k$
- ④ Free to adjust the height of T_k
- ⑤ $mE_k(T_k)$ arbitrarily larger than $mE_{k+1}(T_k)$.



¹Edge in the induced subtree.

Theorem

Let $C_n = (V, E)$ be a cycle of length n . Then:

- if $n \equiv 0 \pmod{4}$, $mE(C_n) = mE_2(C_n) = \frac{3}{2}|E|$
- if $n \equiv 2 \pmod{4}$, $mE(C_n) = mE_3(C_n) = \frac{3}{2}|E| + 3$
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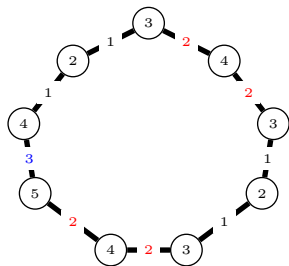
Claim

Let ℓ be a proper k -labelling of $C_n = (V, E)$ and for $\alpha \leq k$, let $S_\alpha = \{e \in E : \ell(e) = \alpha\}$. Then $\forall \alpha \leq k$:

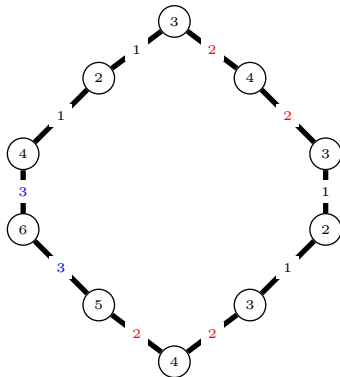
- if $n \equiv 0 \pmod{4}$, $S_\alpha \leq \frac{1}{2}|E|$
- if $n \equiv 2 \pmod{4}$, $S_\alpha \leq \frac{1}{2}|E| - 1$
- otherwise, $S_\alpha \leq \lfloor \frac{1}{2}|E| \rfloor$

Cycles

Provide labelling that reaches the lower bound. Examples:



$$mE(C_9) = 15$$



$$mE(C_{10}) = 18$$

Trees

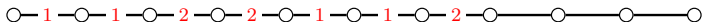
$$mE_2(T) \leq \frac{3}{2}|E|$$

Theorem

Let $T = (V, E)$ be a tree. Then $mE_2(T) \leq \frac{3}{2}|E|$.

Proof: induction on number of branching nodes (i.e. vertices with degree ≥ 3).

Base: T is a path.



T

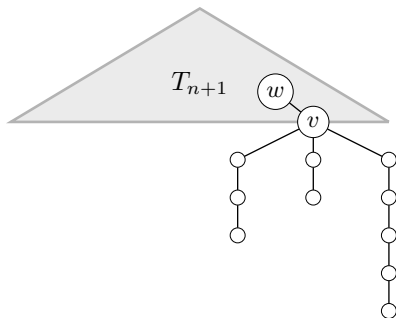
Observation: Edges at distance two, different label

Half edges labelled 1, at most half edges labelled 2 $\Rightarrow mE_2(T) \leq \frac{3}{2}|E|$

$$mE_2(T) \leq \frac{3}{2}|E|$$

T_{n+1} : rooted at a branching node.

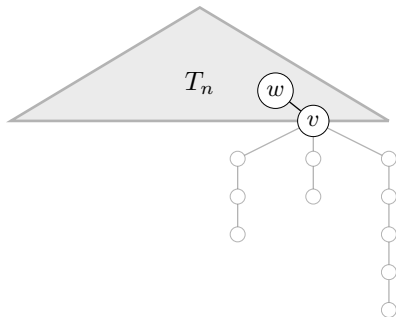
Induction on number of branching nodes:



$$mE_2(T) \leq \frac{3}{2}|E|$$

T_n : rooted at a branching node.

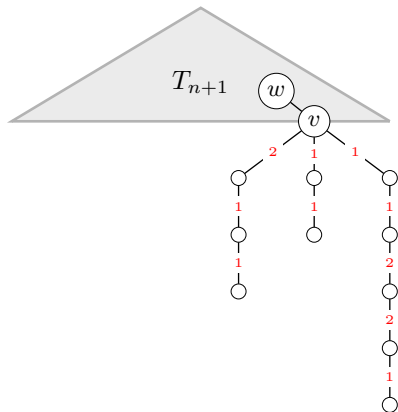
Induction on number of branching nodes:



$$mE_2(T) \leq \frac{3}{2}|E|$$

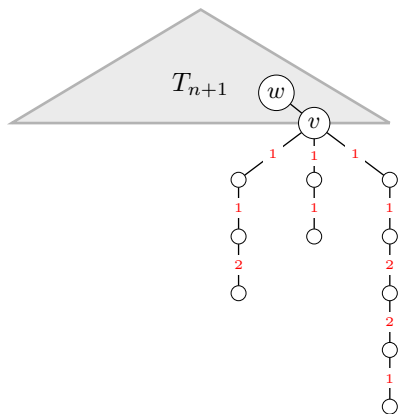
T_{n+1} : rooted at a branching node.

Induction on number of branching nodes:



Add them back using 1, 1, 2, 2, 1, ... or 2, 1, 1, 2, 2, 1, ...

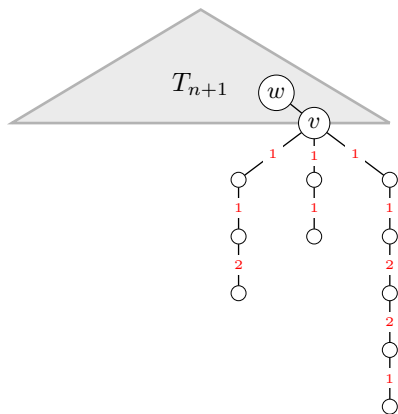
$$mE_2(T) \leq \frac{3}{2}|E|$$



Add them back using $1, 1, 2, 2, 1, \dots$ or $2, 1, 1, 2, 2, 1, \dots$

- No conflict between v and its children (different degrees)

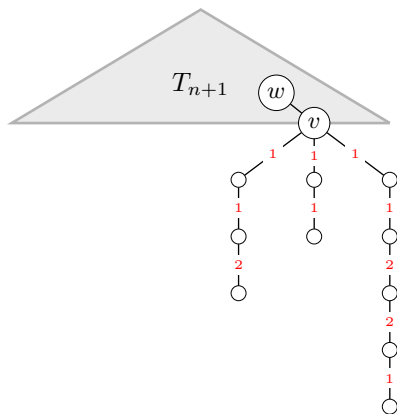
$$mE_2(T) \leq \frac{3}{2}|E|$$



Add them back using $1, 1, 2, 2, 1, \dots$ or $2, 1, 1, 2, 2, 1, \dots$

- No conflict between v and its children (different degrees)
- Assign labels $1, 1, 2, 2, \dots$ on all added paths

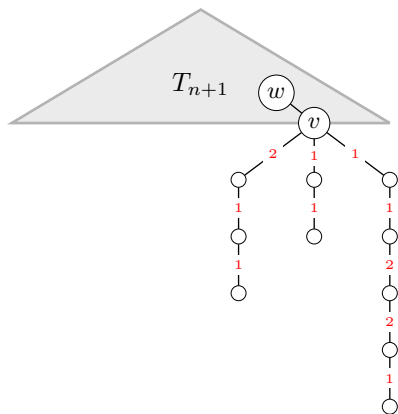
$$mE_2(T) \leq \frac{3}{2}|E|$$



Add them back using 1, 1, 2, 2, 1, ... or 2, 1, 1, 2, 2, 1, ...

- No conflict between v and its children (different degrees)
- Assign labels 1, 1, 2, 2, ... on all added paths
- If $c_\ell(w) = c_\ell(v)$, find a path of length $\not\equiv 1 \pmod{4}$ and label 2, 1, 1, 2, 2, 1, ...

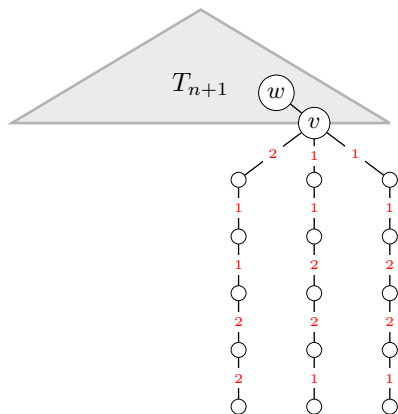
$$mE_2(T) \leq \frac{3}{2}|E|$$



Add them back using $1, 1, 2, 2, 1, \dots$ or $2, 1, 1, 2, 2, 1, \dots$

- No conflict between v and its children (different degrees)
- Assign labels $1, 1, 2, 2, \dots$ on all added paths
- If $c_\ell(w) = c_\ell(v)$, find a path of length $\not\equiv 1 \pmod{4}$ and label $2, 1, 1, 2, 2, 1, \dots$

$$mE_2(T) \leq \frac{3}{2}|E|$$



Add them back using $1, 1, 2, 2, 1, \dots$ or $2, 1, 1, 2, 2, 1, \dots$

- No conflict between v and its children (different degrees)
- Assign labels $1, 1, 2, 2, \dots$ on all added paths
- If $c_\ell(w) = c_\ell(v)$ and all paths of length $\equiv 1 \pmod{4}$ label $2, 1, 1, 2, 2, 1, \dots$ on random path.